2. DEFINITIONS

2.1 Definition of the Feynman rules

The purpose of this section is to spell out the precise form of the Feynman rules for a given Lagrangian. In principle, this is very straightforward: the propagators are defined by the quadratic part of the Lagrangian, and the rest is represented by vertices. As is well known, the propagators are minus the inverse of the operator found in the quadratic term, for example

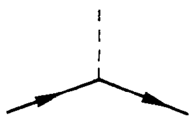
$$\mathcal{L} = \frac{1}{2}\,\phi(\partial^2 - m^2)\phi \;\rightarrow\; (k^2 + m^2 - i\varepsilon)^{-1} \;,$$

$$\mathcal{L} = -\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi \;\rightarrow\; (i\gamma k + m)^{-1} = \frac{-i\gamma k + m}{k^2 + m^2 - i\varepsilon} \;.$$

Customarily, one derives this using commutation rules of the fields, etc. We will simply skip the derivation and define the propagator, including the $i\varepsilon$ prescription for the pole.

Similarly, vertices arise. For instance, if the interaction Lagrangian contains a term providing for the interaction of fermions and a scalar field one has

$$\mathcal{L}_I = g(\bar{\psi}\psi)\phi$$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In this and similar cases there is no difficulty in deriving the rules by the usual canonical formalism. If however derivatives, or worse non-local terms, occur in $\mathcal{L}$, then complications arise. Again we will short-circuit all difficulties and define our vertices, including non-local vertices, directly from the Lagrangian. Furthermore, we will allow sources that can absorb or emit particles. They are an important tool in the analysis. In the rest of this section we will try to define precisely the Feynman rules for the general case, including factors π, etc. Basically the recipe is the straightforward generalization of the simple cases shown above.

The most general Lagrangian to be discussed here is

$$\mathcal{L}(x) = \psi_i^*(x)V_{ij}\psi_j(x) + \frac{1}{2}\,\phi_i(x)W_{ij}\phi_j(x) + \mathcal{L}_I(\psi^*, \psi, \phi) \;. \tag{2.1}$$

The ψ_i and ϕ_i denote sets of complex and real fields that may be scalar, spinor, vector, tensor, etc., fields. The index i stands for any spinor, Lorentz, isospin, etc., index. V and W are matrix operators that may contain derivatives, and whose Fourier transform must have an inverse. Furthermore, these inverses must satisfy the Källén-Lehmann representation, to be discussed later. The interaction Lagrangian $\mathcal{L}_I(\psi^*,\psi,\phi)$ is any polynomial in certain coupling constants g as well as the fields. This interaction Lagrangian is allowed to be non-local, i.e. not only depend on fields in the point x, but also on fields at other space time points x', x'', ... The coefficients in the polynomial expansion may be functions of x. The explicit form of a general term in $\mathcal{L}_I(x)$ is

$$\int d_4x_1\, d_4x_2,\, \ldots,\, \alpha_{i_1 i_2 \ldots}(x, x_1, x_2, \ldots)$$
$$\psi^*_{i_1}(x_1),\, \ldots,\, \psi_{i_m}(x_m),\, \ldots,\, \phi_{i_n}(x_n),\, \ldots\ . \tag{2.2}$$

The α may contain any number of differential operators working on the various fields.

Roughly speaking propagators are defined to be minus the inverse of the Fourier transforms of V and W, and vertices as the Fourier transforms of the coefficients α in $\mathcal{L}_I$.

The action S is defined by

$$iS = i \int d_4x\, \mathcal{L}(x)\ . \tag{2.3}$$

In $\mathcal{L}$ we make the replacement

$$\psi_i(x) = \int d_4k\ a_i(k)e^{ikx}\ ,$$

$$\psi^*_i(x) = \int d_4k\ b_i(k)e^{ikx}\ ,$$

$$\phi_i(x) = \int d_4k\ c_i(k)e^{ikx}\ ,$$

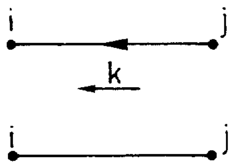
$$\alpha_{i_1 i_2 \ldots}(x, x_1, x_2, \ldots) =$$
$$= \int d_4k\, d_4k_1\, d_4k_2,\, \ldots,\, e^{ikx+ik_1(x-x_1)+ik_2(x-x_2)+\ldots}\ \bar{\alpha}_{i_1\ldots}(k, k_1, k_2, \ldots)\ .$$

The action times i takes the form

$$(2\pi)^4 ib(k)\bar{V}_{ij}(k)a_j(k) + \frac{1}{2}(2\pi)^4 ic_i(k)\bar{W}_{ij}(k)c_j(k)$$
$$+ \ldots + (2\pi)^4 i\delta_4(k + k_1 + \ldots)\bar{\alpha}_i(k, k_1, k_2, \ldots) \times$$
$$\times\, b_{i_1}(k_1),\, \ldots,\, a_{i_m}(k_m),\, \ldots,\, c_{i_n}(k_n) + \ldots\ , \tag{2.4}$$

each term integrated over the momenta involved. The $\bar{V}$ and $\bar{W}$ contain a factor ik_n (or $-ik_n$) for every derivative $\partial/\partial x_\mu$ acting to the right (left) in V and W, respectively. The $\bar{\alpha}$ contain a factor $ik_{j\mu}$ for every derivative $\partial/\partial x_{j\mu}$ acting on a field with argument x_j.

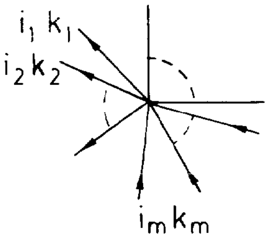
The propagators are defined to be:



$$\Delta_{Fij}(k) = -\frac{1}{(2\pi)^4 i}\left[\bar{V}^{-1}(k)\right]_{ij}\ ,$$
$$\Delta_{Fij}(k) = -\frac{1}{(2\pi)^4 i}\left[\frac{1}{2}\bar{W}(k) + \frac{1}{2}\tilde{\bar{W}}(-k)\right]^{-1}_{ij}. \tag{2.5}$$

Here $\tilde{\bar{W}}$ is $\bar{W}$ reflected, i.e. $\tilde{\bar{W}}_{ij} = \bar{W}_{ji}$. In the rare case of real fermions the propagator must be minus the inverse of the antisymmetric part of W. Furthermore, there is the usual $i\varepsilon$ prescription for the poles of these propagators. The momentum k in Eqs. (2.5) is the momentum flow in the direction of the arrow.

The definition of the vertices is:



$$(2\pi)^4 i \sum_{\{1\ldots m-1\}} \sum_{\{m\ldots n-1\}} \sum_{\{n\ldots\}} (-1)^P \times$$

$$\times\, \alpha_{i_1\ldots}(k, k_1, k_2, \ldots)\delta_4(k + k_1 + \ldots\ ) \ . \quad (2.6)$$

The summation is over all permutations of the indices and momenta indicated. The momenta are taken to flow inwards. Any field ψ^* corresponds to a line with an arrow pointing outwards; a field ψ gives an opposite arrow. The ϕ fields give arrow-less lines. The factor $(-1)^P$ is only of importance if several fermion fields occur. All fermion fields are taken to anticommute with all other fermion fields. There is a factor -1 for every permutation exchanging two fermion fields.

The coefficients α will often be constants. Then the sum over permutations results simply in a factor. It is convenient to include such factors already in $\mathcal{L}_I$; for instance

$$\mathcal{L}_I(x) = \frac{1}{3!\,6!}\, g\psi^*(x)^3\psi(x)^6$$

gives as vertex simply the constant g.

As indicated, the coefficients α may be functions of x, corresponding to some arbitrary dependence on the momentum k in (2.6). This momentum is not associated with any of the lines of the vertex. If we have such a k dependence, i.e. the coefficient α is non-zero for some non-zero value of k, then this vertex will be called a source. Sources will be indicated by a cross or other convenient notation as the need arises.

A diagram is obtained by connecting vertices and sources by means of propagators in accordance with the arrow notations. Any diagram is provided with a combinatorial factor that corrects for double counting in case identical particles occur. The computation of these factors is somewhat cumbersome; the recipe is given in Appendix A.

Further, if fermions occur, diagrams are provided with a sign. The rule is as follows:

i) there is a minus sign for every closed fermion loop;
ii) diagrams that are related to each other by the omission or addition of boson lines have the same sign;
iii) diagrams related by the exchange of two fermion lines, internal or external, have a relative minus sign.