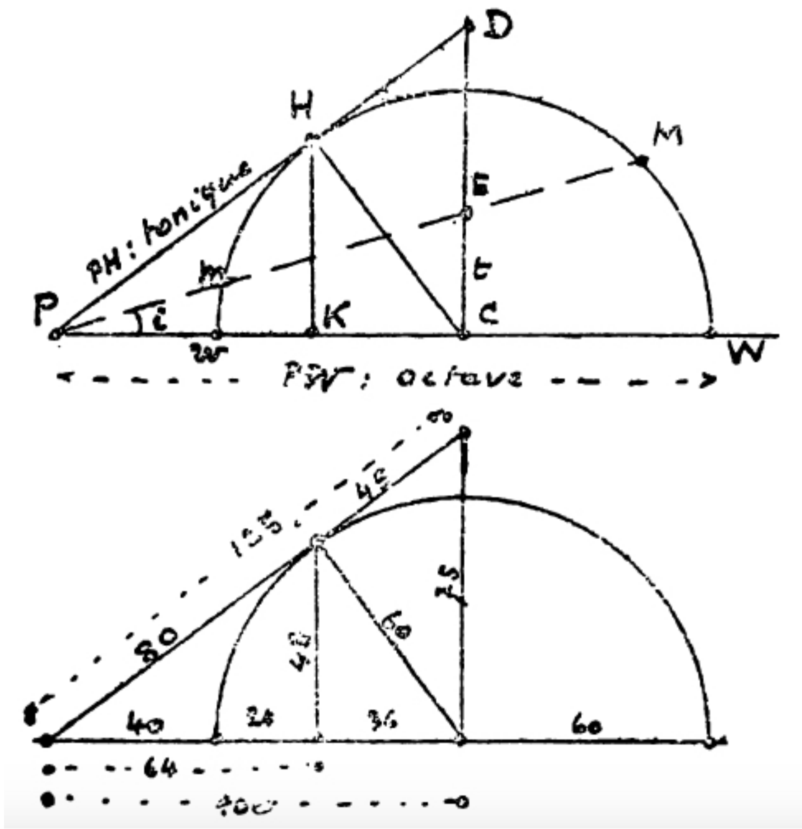
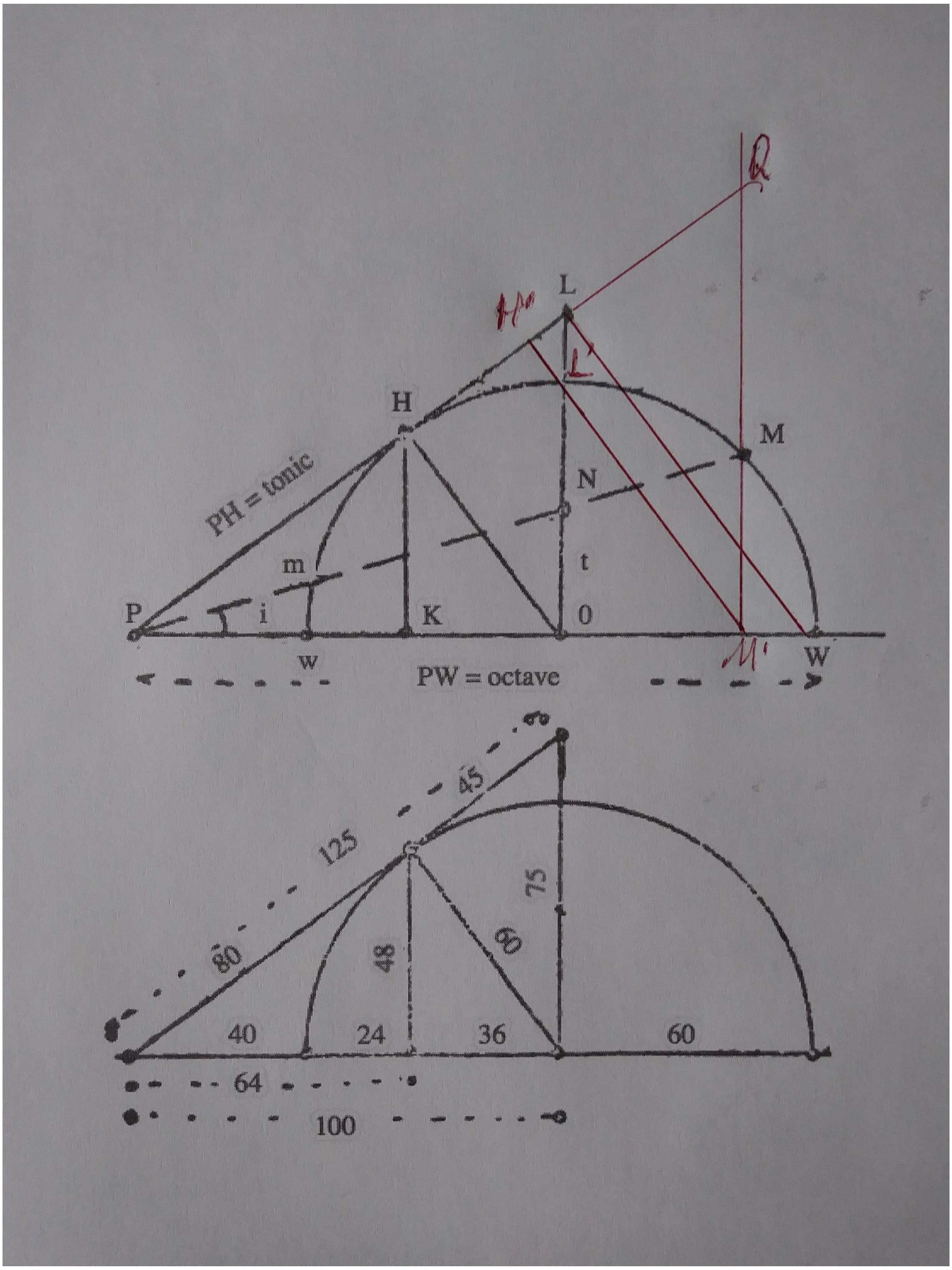
Hello, everybody:

I am a philologist who is fond of mathematics, but who unfortunately has just an elementary high school knowledge of them. I am translating *La leçon de Platon,* by Dom Néroman (La Bégude de Mazenc, Arma Artis, 2002), which deals with music theory and mathematics in the works of Plato. The problem which brings me here is not about translation, but about mathematics. It is a long and complex one, so please take your time for examining it.

On p. 93 of that book, there is this figure:



In the reproduction of this image in my translation, I have not only translated the indications in French, but also substituted 0 for C, and L for D, to avoid any confusion with C and D as names of musical notes in English. For dealing with the problem that I am going to expose here, I have also added some lines and letters in red ink:



On p. 91-2, the author gives a series of measurements of segments of that figure, assuming that PH = 1; PW = 2, and PW x Pw = 2 (2 – wW) = 1. Then, Pw = PW – wW = 2 – wW; PW x Pw = 2 (2 – wW) = 1; 4 – 2wW = 1; 4 – 1 = 2wW; 3 = 2wW; wW = 3/2. So, 0H = (wW)/2 = (3/2)/2 = 3/4, and P0 = $\sqrt{1 + \frac{9}{16}} = \frac{5}{4}$.

Here we have the values of all the lengths of the above construction:

PH = h (base).	For h = 80
0H = radius of the circle = $\frac{3}{4}h$;	0H = 60
P0 = $\frac{5}{4}h$;	P0 = 100
PW = $2h$;	PW = 160
Pw = $\frac{1}{2}h$;[1]	Pw = 40
PK = $\frac{4}{5}h$;[2]	PK = 64
HK = $\frac{3}{5}h$;[3]	HK = 48
K0 = $\frac{9}{20}h$;[4]	K0 = 36
0L = $\frac{15}{16}h$;[5]	0L = 75

---

[1] Since $Pw = P0 - \frac{wW}{2} = \frac{5}{4} - \frac{3}{4} = \frac{1}{2}$.

[2] We must find PK, such that PK + K0 = P0 = 5/4, being 0H = 3/4, and PH = 1. Then, we shall have:

a) $(PK + K0)^2 = PH^2 + 0H^2 \leftrightarrow PK^2 + 2PK \cdot K0 + K0^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$

b) On the other hand, according to the theorem of the cathetus being proportional mean between the hypotenuse and the projection of the cathetus on the hypotenuse, H0 is proportional mean between P0 and K0, that is:

$$\frac{H0}{P0} = \frac{K0}{H0}.$$

c) Now, since we know that P0 = 5/4, and H0 = 3/4, we may find K0:

$$\frac{\frac{3}{4}}{\frac{5}{4}} = \frac{K0}{\frac{3}{4}} \leftrightarrow K0 = \frac{\left(\frac{3}{4}\right)^2}{\frac{5}{4}} = \frac{\frac{9}{16}}{\frac{5}{4}} = \frac{36}{80} = \frac{9}{20}.$$

d) And last, since PK + K0 = P0 = 5/4,
PK + (9/20) = 5/4; PK = (5/4) – (9/20) = (25/20) – (9/20) = 16/20 = 4/5.

[3] We already know that PK = 4/5 and PH = 1. Then, 1 = $(4/5)^2$ + HK^2, and $HK = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{25}{25} - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

[4] See above, n. 2.

[5] a) First of all, the triangles 0HL and P0L being similar, their corresponding sides are proportional. We must find 0L, such that $PL^2 = 0L^2 + P0^2$, where we only know that P0 = 5/4; but, on the other hand:

b) We know PH = 1, and PL = PH + HL = 1 + HL, so that

$$PL^2 = 0L^2 + P0^2 \leftrightarrow (1 + HL)^2 = 0L^2 + \left(\frac{5}{4}\right)^2$$

c) We know that P0 = 5/4, is proportional mean between the hypotenuse (PL, unknown) and the projection of P0 on the hypotenuse (PH = 1), so that

$$\frac{P0}{PL} = \frac{PH}{P0} \leftrightarrow \frac{\frac{5}{4}}{PL} = \frac{1}{\frac{5}{4}} \leftrightarrow \frac{5}{4} = \frac{PL}{\frac{5}{4}} \leftrightarrow PL = \frac{25}{16},$$

and, since

$$PL = PH + HL,$$

$$\frac{25}{16} = 1 + HL \leftrightarrow HL = \frac{25}{16} - 1 = \frac{9}{16}.$$

d) Now we must find 0L, by means of the theorem of Pythagoras:

$$PL^2 = P0^2 + 0L^2 \leftrightarrow \left(\frac{25}{16}\right)^2 = \frac{25}{16} + 0L^2 \leftrightarrow 0L^2 = \left(\frac{25}{16}\right)^2 - \frac{25}{16} = \frac{625}{256} - \frac{25}{16} = \frac{625 - 400}{256} = \frac{225}{256} \leftrightarrow 0L = \sqrt{\frac{225}{256}} = \frac{15}{16}.$$

HL = $\frac{9}{16}h$, as we saw above. HL = 45

$\frac{0\text{L}}{\text{P}0}$ = tangent of the interval of octave = $\frac{3}{4}$. He means that 0L / P0 is the tangent of the angle P: 0L / P0 = (15/16) / (5/4) = 60 / 80 = 3/4.

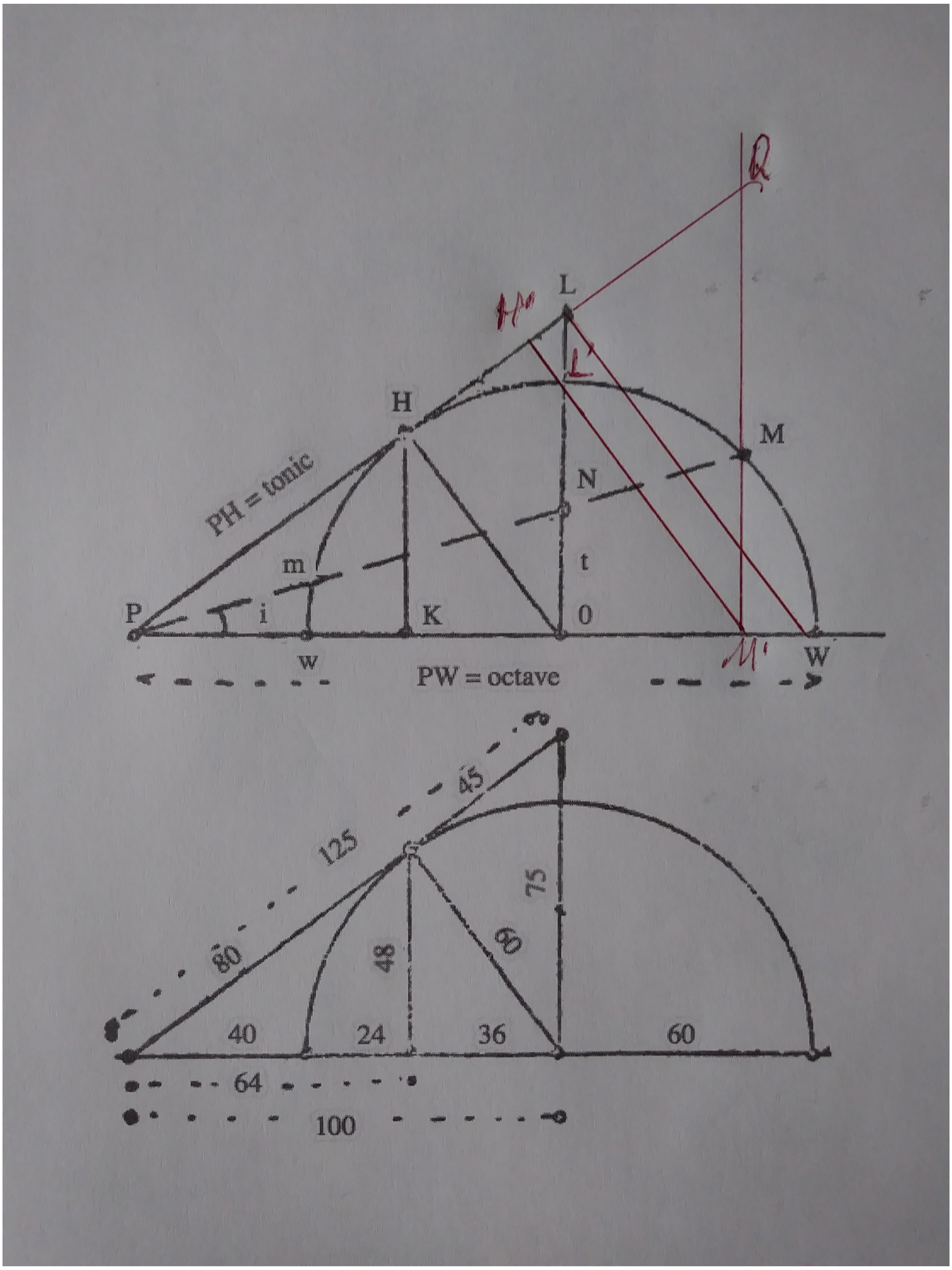
$t = \frac{5}{4}h \cdot \tan i$.

The author gives the numerical values for h = 80, a value which allows to express all in integers.

Then, on p. 93, the author says that, if we take the letter *f* for the ratio $\frac{PM}{PH}$, it is easy to demonstrate that $f = 5 \pm \frac{\sqrt{9-16(\tan i)^2}}{4\sqrt{1+(\tan i)^2}}$. But he does not demonstrate it. This is my problem. I would like to demonstrate that myself, but I have not been able to do that so far.

I have thought that I must express PM and PH as functions of tan *i:*

a) I think it is not too complicated to express PM as a function of tan *i*. Let us remember how I completed the figure as a helping device:



For expressing PM as a function of tan i, taking into account that $\tan i = \frac{MM\prime}{PM\prime}$, we may do this:

$\tan i = \frac{MM\prime}{PM\prime} \leftrightarrow MM' = PM' \cdot \tan i\,;\; PM^2 = (PM' \cdot \tan i)^2 + PM'^2 = PM'^2[1 + (\tan i)^2] \leftrightarrow PM = \sqrt{PM'^2[1 + (\tan i)^2]} = PM'\sqrt{[1 + (\tan i)^2]}$

b) For expressing PH as a function of tan *i,* I have noticed that, if we draw a perpendicular to PL, passing by the point in which 0L cuts the semi-circumference of our figure, and we lengthen it until it cuts the segment wW, we obtain the point M' of the segments PM' and MM' with which we operated in the previous section. This may help us, as we shall see later.

On the other hand, there are two triangles, PH'M' and PLW, which are similar to PH0.

We know the dimensions of PH0: PH = 1; 0H = 3/4, and P0 = 5/4.

On the other hand, 0H = 0w = 0W, so that PW = P0 + 0w = P0 + 0H = (5/4) + (3/4) = 8/4 = 2. This may help to find LW, since $\frac{P0}{PW} = \frac{H0}{LW} \leftrightarrow \frac{\frac{5}{4}}{2} = \frac{\frac{3}{4}}{LW} \leftrightarrow \frac{LW \cdot \frac{5}{4}}{2} = \frac{3}{4} \leftrightarrow LW \cdot \frac{5}{4} = 2 \cdot \frac{3}{4} = \frac{6}{4} \leftrightarrow LW = \frac{\frac{6}{4}}{\frac{5}{4}} = \frac{24}{20} = \frac{6}{5}$.

Now we must find 0L. The triangles PHK and PL0 being similar,

$$\frac{PK}{P0} = \frac{HK}{0L} \leftrightarrow \frac{\frac{4}{5}}{\frac{5}{4}} = \frac{\frac{3}{5}}{0L} \leftrightarrow \frac{16}{25} = \frac{\frac{3}{5}}{0L} \leftrightarrow \frac{16}{25} \cdot 0L = \frac{3}{5} \leftrightarrow 0L = \frac{\frac{3}{5}}{\frac{16}{25}} = \frac{75}{80} = \frac{15}{16}.$$

Now, knowing that 0L = 15/16, that 0L' = 0H = 0W = 3/4, and that the triangles W0L and M'0L' are similar, we may determine 0M':

$$\frac{0L'}{0L} = \frac{0M'}{0W} \leftrightarrow \frac{\frac{3}{4}}{\frac{15}{16}} = \frac{0M'}{\frac{3}{4}} \leftrightarrow 0M' = \frac{\frac{9}{16}}{\frac{15}{16}} = \frac{144}{240} = \frac{3}{5}.$$

Our next step may be to determine MM'. We already know 0M' = 3/5, and 0M = 0H = 0W = 3/4. So, according to the theorem of Pythagoras:

$$0M^2 = 0M'^2 + MM'^2 \leftrightarrow MM' = \sqrt{0M^2 - 0M'^2} = \sqrt{\left(\frac{3}{4}\right)^2 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{9}{16} - \frac{9}{25}}$$

$$= \sqrt{\frac{225 - 144}{400}} = \sqrt{\frac{81}{400}} = \frac{9}{20}.$$

Now, once we know PM', we must find the values of the other cathetus and of the hypotenuse of PM'Q, similar to PHK and to P0L. PQ is a prolongation of PH, and QM' = QM + MM'. So:

$$\frac{PK}{PM'} = \frac{HK}{QM'} \leftrightarrow \frac{\frac{4}{5}}{P0 + 0M'} = \frac{\frac{3}{5}}{QM'} \leftrightarrow \frac{\frac{4}{5}}{\frac{5}{4} + \frac{3}{5}} = \frac{\frac{3}{5}}{QM'} \leftrightarrow \frac{\frac{4}{5}}{\frac{25+12}{20}} = \frac{\frac{3}{5}}{QM'} \leftrightarrow \frac{\frac{4}{5}}{\frac{37}{20}}$$

$$= \frac{\frac{3}{5}}{QM'} \leftrightarrow \frac{80}{185} = \frac{\frac{3}{5}}{QM'} \leftrightarrow QM' \cdot \frac{80}{185} = \frac{3}{5} \leftrightarrow QM' = \frac{\frac{3}{5}}{\frac{80}{185}} = \frac{555}{400}$$

$$= \frac{111}{80}.$$

We must still find the value of PQ:

$$\frac{PH}{PQ} = \frac{PK}{PM'} \leftrightarrow \frac{1}{PQ} = \frac{\frac{4}{5}}{\frac{37}{20}} \leftrightarrow 1 = PQ \cdot \frac{\frac{4}{5}}{\frac{37}{20}} \leftrightarrow PQ = \frac{\frac{37}{20}}{\frac{4}{5}} = \frac{185}{80} = \frac{37}{16}.$$

Before we proceed, we may sum up:

a) First, PH = 1; PQ = (37/16); PM' = (37/20); MM' = 9/20, and QM' = 111/80. Thence, we may already deduce that PQ – PH = (37/16) – (16/16) = (21/16), that is, PQ = 1 + (21/16).

b) On the other hand, $\tan i = \frac{MM'}{PM'} \leftrightarrow PM' = \frac{MM'}{\tan i}$ y $MM' = PM' \cdot \tan i$.

c) QM' = (111/80), and MM' = (9/20), so that QM' = (9/20) + QM, that is, QM = (111/80) – (9/20) = (111/80) – (36/80) = (75/80), or QM' = [(75/80) + MM'] = [(75/80) + (PM' x tan *i*)].

Now, taking all that into consideration, and according to the theorem of Pythagoras:

$PQ^2 = PM'^2 + QM'^2 \leftrightarrow \left(PH + \frac{21}{16}\right)^2 = \left(\frac{MM'}{\tan i}\right)^2 + \left[\frac{75}{80} + (PM' \cdot \tan i)\right]^2 \leftrightarrow \left[PH^2 + \left(2 \cdot \frac{21}{16} \cdot PH\right) + \left(\frac{21}{16}\right)^2\right] = \left(\frac{MM'}{\tan i}\right)^2 + \left[\left(\frac{75}{80}\right)^2 + \left(2 \cdot \frac{75}{80} \cdot PM' \cdot \tan i\right) + PM'^2 \cdot (\tan i)^2\right] \leftrightarrow$

$PH^2 + \frac{42}{16}PH + \left[\left(\frac{21}{16}\right)^2 - \left(\frac{MM'}{\tan i}\right)^2 - \left(\frac{75}{80}\right)^2 - \left(2 \cdot \frac{75}{80} \cdot PM' \cdot \tan i\right) - (PM' \cdot \tan i)^2\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \left[\frac{441}{256} - \frac{5625}{6400} - \left(\frac{MM'}{\tan i}\right)^2 - \left(2 \cdot \frac{75}{80} \cdot PM' \cdot \tan i\right) - (PM' \cdot \tan i)^2\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \left[\frac{11025-5625}{6400} - \frac{MM'^2}{(\tan i)^2} - \left(2 \cdot \frac{75}{80} \cdot PM' \cdot \tan i\right) - (PM' \cdot \tan i)^2\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \left[\frac{5400}{6400} - \frac{MM'^2 - \left(2 \cdot \frac{75}{80} \cdot PM' \cdot \{\tan i\}^3\right) - PM' \cdot (\tan i)^4}{(\tan i)^2}\right] = 0.$

Now we may proceed as follows:

$$PH^2 + \frac{42}{16}PH + \left[\frac{5400}{6400} - \frac{MM'^2 - \left(2\cdot\frac{75}{80}\cdot PM'\cdot\{\tan i\}^3\right) - PM'\cdot(\tan i)^4}{(\tan i)^2}\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \left[\frac{5400}{6400} - \frac{\{PM'^2\cdot(\tan i)^2\} - PM'\cdot\left(\left\{\frac{150}{80}\cdot[\tan i]^3\right\} + \{\tan i\}^4\right)}{(\tan i)^2}\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \left[\frac{5400}{6400} - \frac{\{PM'\cdot(\tan i)^2\}\cdot\left\{PM' - \left(\frac{150}{80}\cdot\tan i\right) - \tan i^2\right\}}{(\tan i)^2}\right] = 0 \leftrightarrow PH^2 + \frac{42}{16}PH + \frac{\{5400\cdot(\tan i)^2\} - 6400\cdot\{PM'\cdot(\tan i)^2\}\cdot\left\{PM' - \left(\frac{150}{80}\cdot\tan i\right) - \tan i^2\right\}}{6400\cdot(\tan i)^2} = 0 \leftrightarrow PH = \frac{-\frac{42}{16} \pm \sqrt{\left(\frac{42}{16}\right)^2 - 4\cdot\left[\frac{\{5400\cdot(\tan i)^2\} - 6400\cdot\{PM'\cdot(\tan i)^2\}\cdot\left\{PM' - \left(\frac{150}{80}\cdot\tan i\right) - \tan i^2\right\}}{6400\cdot(\tan i)^2}\right]}}{2}.$$

There we have PH expressed as a function of tan *i* and PM’. If we remember that $PM = PM'\sqrt{[1 + (\tan i)^2]}$, we may write:

$$\frac{PM}{PH} = \frac{PM'\sqrt{[1 + (\tan i)^2]}}{\frac{-\frac{42}{16} \pm \sqrt{\left(\frac{42}{16}\right)^2 - 4\cdot\left[\frac{\{5400\cdot(\tan i)^2\} - 6400\cdot\{PM'\cdot(\tan i)^2\}\cdot\left\{PM' - \left(\frac{150}{80}\cdot\tan i\right) - \tan i^2\right\}}{6400\cdot(\tan i)^2}\right]}}{2}} = \frac{2\cdot PM'\cdot\sqrt{[1 + (\tan i)^2]}}{-\frac{42}{16} \pm \sqrt{\left(\frac{42}{16}\right)^2 - 4\cdot\left[\frac{\{5400\cdot(\tan i)^2\} - 6400\cdot\{PM'\cdot(\tan i)^2\}\cdot\left\{PM' - \left(\frac{150}{80}\cdot\tan i\right) - \tan i^2\right\}}{6400\cdot(\tan i)^2}\right]}}$$

But I cannot see how to simplify that monstrous equation to obtain $\frac{PM}{PH} = 5 \pm \frac{\sqrt{9 - 16(\tan i)^2}}{4\sqrt{1 + (\tan i)^2}}$.

Thank you very much in advance for whatever help in detecting any wrong assumptions or calculations, and in the demonstration of that formula.

All best!