Using this result for $x_0 = x_1$, $y_0 = y_1$ and putting $\lambda = -\mathrm{i}g$ again, (6) becomes

$$\tilde{P}(g, N) = \frac{\frac{1}{4}Ngl^2}{\sinh(\frac{1}{4}Ngl^2)}. \tag{20}$$

Note that $\tilde{P}(0, N) = 1$, as it should because the integral of the distribution function for A should be normalised to unity. Substituting the previous equation into (5) one finds that $P(A, N)$ is of the form

$$P(A, N) = \frac{2}{\pi N l^2} p\left(\frac{4A}{Nl^2}\right) \tag{21}$$

where

$$p(\xi) = \int_{-\infty}^{+\infty} \frac{u}{\sinh u} \exp(\mathrm{i}u\xi)\, \mathrm{d}u. \tag{22}$$

The integrand has singularities at $u = n\pi\mathrm{i}$ with $n = \pm 1, \pm 2, \ldots$. Near $n\pi\mathrm{i}$ one can write $\sinh u \simeq (-1)^n(u - n\pi\mathrm{i})$. For a positive value of ξ the contour of integration can be closed with a semicircle in the positive complex u plane. Hence Cauchy's theorem gives a series for $p(\xi)$ which can be summed to give

$$p(\xi) = \pi/4\cosh^2(\tfrac{1}{2}\pi\xi). \tag{23}$$

Combination of (21) and (23) gives the final result

$$P(A, N) = \left[2Nl^2 \cosh^2\left(\frac{2\pi A}{Nl^2}\right)\right]^{-1}. \tag{24}$$

This analytical result for the continuous random walk model is remarkably simple. In order to compare with the results of Brereton and Butler (1987) for the discrete model we should consider $P(A, N)$ as a function of A/Nl^2 for large N. Equation (24) starts with the value $\frac{1}{2}$ for $(A/Nl^2) = 0$ and has the asymptotic behaviour $\exp(-4\pi A/Nl^2)$ for $(A/Nl^2) \gg 1$. The results for $P(A, N)$ for the discrete model start with a value of about 0.53 and have the asymptotic behaviour $\exp(-aA/bNl^2)$ with $a \simeq 4.95$ and $b \simeq 0.45$. As $a/b \simeq 10.9$ is fairly close to $4\pi \simeq 12.6$ we conclude that the continuous random walk result (24) is a fair approximation to the distribution of the area enclosed by a discrete random walk.

References

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