The Electromagnetic Power Flux Revisited

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ABSTRACT

We examine the common conceptions regarding the electromagnetic energy, as well as the ambiguity of its flux density. An alernative form of the flux will be proposed, which better fits with the common intuition for steady currents in electrical wires, and provide a rigorous basis for the electrochemical potential energy flux in thermodynamics. Of course, the alternative form is compatible with Maxwell equations and the energy density formula.

1. Electromagnetic Energy density and power flow

The expression of the energy density of the electromagnetic field was first derived in 1880 by J. C. Maxwell ([Ma]) in his monumental work. Maxwell showed essentially that the (volumic) density of E.M. energy at some point M is equal to

$$u = \frac{\varepsilon_0}{2}\mathbf{E^2} + \frac{\varepsilon_0 c^2}{2}\mathbf{B^2}, \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields resp. at M, ε_0 is the permitivity of free space, and c is the speed of light.

This expression fitted well with the once new theory of continuous propagation of the electrical and magnetic fields, advocated by Faraday and Maxwell, in contrast to the theory of action at some distance developed by the German physisists.

Relation (1) is now considered as fundamental, and is firmly anchored into the theory of General relativity ([Wa], Sec. 1.2 p. 6).

Building on the work of Maxwell, J. H. Poynting introduced in 1883 the vector that bears his name ([Po]). The Poynting vector is defined at any point M by

$$\mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B}. \tag{2}$$

It can be shown, using Maxwell's equations, that $\mathbf{S}$ fulfills

$$-\frac{\partial u}{\partial t} = \nabla \cdot S + \mathbf{E} \cdot \mathbf{j}, \tag{3}$$

where $\mathbf{j}$ is the current density at M. Since $\mathbf{E} \cdot \mathbf{j}$ is the work done by the E.M. field, or more precisely, the power transmitted to the charge element $q = \rho dV$ centered at

M, this formula, when integrated on a closed volume $\mathcal{V}$ whose surface is oriented to the outer side, means that the rate of decrease of the E.M. energy contained inside a volume $\mathcal{V}$, or what is the same, the E.M. power wasted by $\mathcal{V}$, is equal to the sum of the flux of $\mathbf{S}$ through the surface ∂V of $\mathcal{V}$, with the power transmitted by the E.M. field to the charges inside $\mathcal{V}$.

This leads to the interpretation of $\mathbf{S}$ as the *power flow density* of the E.M. field, flowing out of $\mathcal{V}$ through its surface.

This interpretation has proved to work well in many circumstances, and to quote Feynmann "nobody has never find anything wrong with it". Nevertheless, and that's the main point of this article, it should be observed that the Poynting vector is not the only possible expression of the power flow density that would give rise a valid theory. Indeed, the only physical way to observe the flow of power seems to be measuring the rate of decrease of the energy inside a closed volume $\mathcal{V}$; in other words, since we have

$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t} - \mathbf{E} \cdot \mathbf{j}, \tag{4}$$

we obtain, integrating over $\mathcal{V}$ with the Stokes formula,

$$\int_{\partial V} \mathbf{S} \cdot \mathrm{d}\vec{s} = -\frac{\partial U}{\partial t} - P, \tag{5}$$

where U is the E.M. energy inside $\mathcal{V}$ and P is the power transmitted to the charges inside $\mathcal{V}$ by the E.M. field. But if some rotational field $\mathbf{R}$ were added to $\mathbf{S}$, say

$$\mathbf{S}' = \mathbf{S} + \mathbf{R},$$

then taking into account that $\nabla \cdot \mathbf{R} = \mathbf{0}$, relation (4) and (5) would hold with $\mathbf{S}'$ in place of $\mathbf{S}$ as well. So, it seems that the real form of the power flow cannot be determined by any physical experiment: power flow is defined only up to a rotational.

Let us report here an entire note of Feynman in his Lectures on Physics ([Fe], II, 27.4):

"Before we take up some applications of the Poynting formulas [Eqs. (27.14) and (27.15)], we would like to say that we have not really proved them. All we did was to find a possible u and a possible $\mathbf{S}$. How do we know that by juggling the terms around some more we couldnt find another formula for u and another formula for $\mathbf{S}$? The new $\mathbf{S}$ and the new u would be different, but they would still satisfy Eq. (27.6). Its possible. It can be done, but the forms that have been found always involve various derivatives of the field (and always with second-order terms like a second derivative or the square of a first derivative). There are, in fact, an infinite number of different possibilities for u and $\mathbf{S}$, and so far no one has thought of an experimental way to tell which one is right! People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain what is the actual location in space of the electromagnetic field energy. So we too will take the easy way out and say that the field energy is given by Eq. (27.14). Then the flow vector $\mathbf{S}$ must be given by Eq. (27.15).

It is interesting that there seems to be no unique way to resolve the indefiniteness in the location of the field energy. It is sometimes claimed that this problem can be resolved by using the theory of gravitation in the following argument. In the theory of gravity, all energy is the source of gravitational attraction. Therefore the energy density of electricity must be located properly if we are to know in which direction the gravity force acts. As yet, however, no one has done such a delicate experiment that the precise location of the gravitational influence of electromagnetic fields could be determined. That electromagnetic fields alone can be the source of gravitational force is an idea it is hard to do without. It has, in fact, been observed that light is deflected as it passes near the sunwe could say that the sun pulls the light down toward it. Do you not want to allow that the light pulls equally on the sun? Anyway, everyone always accepts the simple expressions we have found for the location of electromagnetic energy and its flow. And although sometimes the results obtained from using them seem strange, nobody has ever found anything wrong with them—that is, no disagreement with experiment. So we will follow the rest of the world—besides, we believe that it is probably perfectly right".

As pointed out by Feynman and virtually all textbooks in electromagnetics, the Poynting vector leads to a description of the power flow that does not fit well with the common natural intuition. As a example taken from the article of Poynting ([Po], p. 350–351, let us examine how the power flows flows inside an electrical wire connected to the terminals of a battery, assuming, for the sake of simplicity, that the wire is cylindrical and has a uniform resistance per unit of length.

At a point M inside the wire or at its surface, the electric field $\mathbf{E}$ is parallel to the segment of wire, since the current is flowing along this direction. Furthermore, since we assume steady currents, $\mathbf{E}$ is constant. On the other hand, assuming a point M is located at the surface of the wire, it can be shown, with, e.g. the Biot and Savard law, that the magnetic field $\mathbf{B}$ at M is is orthogonal to the radius OM of the wire, and directed toward its axis: $\mathbf{B} = \alpha \mathbf{E} \times \overrightarrow{OM}$, with $\alpha > 0$. So, the Poynting vector $\mathbf{S} = \varepsilon_0 c^2 \mathbf{E} \times \mathbf{B}$ is radial: $\mathbf{S} = \beta \overrightarrow{OM}$, with $\beta < 0$. This means that according to this theory, the E.M. power is not flowing inside and along the wire, as one would intuitively expect, but it comes from the outside and flows perpendicularly through the surface toward the axis of the wire.

The amount of electrical power transmitted to a length ℓ of the wire can also be easily computed: Let $\mathcal{A}$ be the area of the cylinder formed by the wire, and $\mathcal{A}'$ its area without the cross sections at the extremities: $\mathcal{A}' = 2\pi r \ell$, where r is the radius of the wire. The power transmitted to the length of wire is

$$P = -\int_{\mathcal{A}} \mathbf{S}\,\mathrm{d}\vec{s} = -\int_{\mathcal{A}'} \mathbf{S}\,\mathrm{d}\vec{s} = 2\pi r\ell\|\mathbf{S}\| = 2\pi r\ell\|\mathbf{E}\|\|\mathbf{B}\|\,.$$

But $2\pi r\|\mathbf{B}\|$ is equal to the circulation of the magnetic field around the loop of radius r, hence is equal, according to Ampere's law, to $\frac{I}{\varepsilon_0 c^2}$, where I is the current intensity in the wire. On the other hand, $V = \ell\|\mathbf{E}\|$ is equal to the positive difference of potential between the two extremities of the length of wire. Hence

$$P = VI\,,$$

which is the well known form of the electrical power transmitted to a resistor, which is eventually dissipated in heat.

This example shows that although the power flow stemming from the Poynting vector seems counter-intuitive, the results are coherent when it comes to computing global energy transfers. This is usually the justification given by virtually all textbooks on this topic, something like "Hey, that's weird, but don't worry, just compute with it and that will work eventually." As we shall see, this need not be the case. It is possible to have a definition of the power flow density that fit with our intuition, yet leading to the same global energy transfer results as the classic definition. In fact, it will even be alleged that the alternative form proposed in the next section is more real than the classic one. But to have a more intuitive power flow density, the price we have to pay is some added complexity in the definition. Nevertheless, if one consider the Poynting vector as a mathematical artifice, it is possible to enjoy both worlds: Just use the Poynting vector to compute global energy transfers, since we already know that this leads to the same results as any alternative power flow density. The poynting vector is indeed the simplest density which could be obtained. As a result, it should be used most often when dealing with global E.M. energy transfers, because any other form would give rise the same results anyway, as shown above. Nevertheless, at the local scale, those forms lead to different results and interpretations. We believe that both in practical and in theoretical situations like thermodynamics, where the E.M. energy interract with other flows and with the matter, the alternative definition given in the next section is the most suitable.

2. Alternative form of the power flow density vector

The alternative form proposed in this section stems from a long discussion in the Physics Forum, about the way the electrical power should be flowing through, or outside, an electrical wire traversed by a steady current. While most of the persons stuck to the classic Poynting view, according to which the electrical power is transmitted radially to the wire, the O.P., "Fluidistic", defended the idea that according to thermodynamics, there *must* exist a component of the power flow along the wire. Indeed, starting from the thermodynamic equation

$$\vec{J}_U = T\vec{J}_S + \bar{\mu}\vec{J}_e\,,$$

where $\vec{J}_U$ is the internal energy flow of the system, $\vec{J}_S$ is the entropy flux, $\vec{J}_e$ is the particule flow density and $\bar{\mu}$ is the electrochemical potential of the electrons, Fluidistic shown that the component $T\vec{J}_S$ of the energy flow is radial to the wire, while $\bar{\mu}\vec{J}_e$ is parallel to the wire. After a long discussion, it was observed by the author that since $\bar{\mu}$ is the sum of the chemical energy of the electron with the potential energy of the electron (with respect to the electric potential), and since the chemical energy of an electron inside a conductor is likely to be null or negligible, $\bar{\mu}J_e$ could simply be written $\phi\mathbf{j}$, where $\mathbf{j}$ is the current density. The author then realized that the whole situation could be formulated in a completely classical electrodynamic framework, that will now be described as a motivating example.

Consider again the electrical wire example of Poynting in the previous section. Along the length of wire ℓ, the electrical potential is $\phi(z)$, z being the curvilinear coordinate oriented from the "plus" to the "minus" terminal of the length ℓ. Notice that ϕ is decreasing, and it decreases linearly if the resistance per unit of length is constant. Now, consider a cross section $\mathcal{A}(z)$ of the wire, and let ρ be the volumic flowing charge density inside the wire. At a point M of $\mathcal{A}(z)$, an elementary charge ρdV has a potential electrical energy equal to $\rho\phi(z)dV$. So, we can define an "electrical potential energy flow density" at M by

$$\mathbf{S}' = \rho\phi\mathbf{v} = \phi\mathbf{j}\,,$$

where $\mathbf{v}$ is the speed of the charge at M, and $\mathbf{j}$ is the current density. The amount of power "flowing" through $\mathcal{A}(z)$ is

$$\int_{\mathcal{A}(z)} \phi\mathbf{j}\,\mathrm{d}\vec{s} = I\phi(z)\,,$$

where I is the current intensity through A, assuming that A is oriented in the same direction as the current. Now, consider as above the cylinder made by a length ℓ of wire. Orient the surface $\mathcal{A}$ of the cylinder as usual, according to the outer normal of the surface. Then denoting by $\mathcal{A}(z_1)$ and $\mathcal{A}(z_2)$ the surfaces at the extremities of the cylinder, and assuming $z_1 < z_2$, the potential energy rate flowing out of the cylinder is

$$\int_{\mathcal{A}} \phi\mathbf{j}\,\mathrm{d}\vec{s} = \int_{A(z_2)} \phi\mathbf{j}\,\mathrm{d}\vec{s} - \int_{A(z_1)} \phi\mathbf{j}\,\mathrm{d}\vec{s} = I(\phi(z_2) - \phi(z_1)) = -IV\,,$$

where V is the positive difference of potential between z_1 and z_2. In other words, the electrical power transmited to the wire is IV, as should be the case for a resistor. We see that we have obtained here the same result as Poynting, but in a completely, almost opposite, route. The power is now flowing inside and along the wire, and has no component orthogonal to the wire, which certainly corresponds better to our intuition of the electrical power.

We now state the proposed alternative form of the power flow density. It is likely to be the simplest form that reduces to $\phi\mathbf{j}$ in the case of steady currents. Its derivation will be performed in the next section.

In term of the potential ϕ and the vector potential $\mathbf{A}$ alone, we define

$$\mathbf{S}' = \varepsilon_0 c^2 \left[\phi \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial \mathbf{A}}{\partial t} \right]. \tag{1}$$

In a somewhat more digest form, using Maxwell's equation $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\mathbf{S}' = \varepsilon_0 c^2 \left[\phi \nabla \times \mathbf{B} + \frac{\partial \mathbf{A}}{\partial t} \right]. \tag{2}$$

It is also possible to transform this expression in order to exhibit the current vector density:

$$\mathbf{S}' = \phi \mathbf{j} + \varepsilon_0 \phi \frac{\partial \mathbf{E}}{\partial \mathbf{t}} + \varepsilon_0 c^2 \frac{\partial \mathbf{A}}{\partial \mathbf{t}}. \tag{3}$$

Observe that for steady currents, eq. (3) reduces to

$$\mathbf{S}' = \phi \mathbf{j}.$$

Admittedly, the alternative form looks more complicated than the Poynting vector form. But is it a reason to believe that the Poynting vector is *more real*? In fact, if we believe that the most real notion should be the simplest, it can be alleged that the alternative form is actually the real one. Indeed, it is now well acknoledged that the electric and magnetic fields are not real, or at least less real than the electric and vector potentials. This fact was pointed out by Feynman, and can be found in many books like [Wa], Sec. 1. Now, if we consider ϕ and $\mathbf{A}$ as the real notions, then the Poynting vector becomes

$$\begin{aligned} \mathbf{S} = \varepsilon_0 c^2 \left[(-\nabla \Phi - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}) \times (\nabla \times A) \right] &= -\varepsilon_0 c^2 \left[\nabla \Phi \times (\nabla \times \mathbf{A}) + \frac{\partial \mathbf{A}}{\partial \mathbf{t}} \times (\nabla \times A) \right] \\ &= -\varepsilon_0 c^2 \left[\nabla \cdot (\mathbf{A} \times \nabla \Phi) + \frac{\partial \mathbf{A}}{\partial \mathbf{t}} \times (\nabla \times A) \right] \end{aligned}$$

The reader is invited to judge which one of the expressions of $\mathbf{S}$ and $\mathbf{S}'$ in term of the potentials is the simplest.

3. Derivation of the alternative form of the power flow density

In this section, we shall use the following well known equations of the electromagnetic theory:

$$\mathbf{B} = \nabla \times \mathbf{A}\,, \tag{1}$$

$$\varepsilon_0 c^2 \nabla \times \mathbf{B} = \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial \mathbf{t}}\,, \tag{2}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}\,, \tag{3}$$

where

- $\mathbf{E}$ is the electric field,
- $\mathbf{B}$ is the magnetic field,
- ϕ is the electric potential,
- $\mathbf{A}$ is the magnetic vector potential,
- ε_0 is the electric permitivity of free space,
- c is the speed of light.

With these equations written, we point out that form $(3)_{\text{p.6}}$ of $\mathbf{S}'$ is derived from $(2)_{\text{p.6}}$ using (2).

We shall now proceed to show that

$$\mathbf{S}' = \mathbf{S} + \mathbf{R}, \quad \text{with} \quad \nabla \cdot \mathbf{R} = 0\,. \tag{4}$$

As explained in the previous section, this implies that $\mathbf{S}'$ leads to a power flow that satisfies energy equation $(3)_{\text{p.1}}$.

It suffices to show that $\nabla \cdot \mathbf{S}' = \nabla \cdot \mathbf{S}$. Starting with eq. $(2)_{\text{p.6}}$, we have

$$\nabla \cdot \mathbf{S}' = \varepsilon_0 c^2 \nabla\phi \cdot (\nabla \times \mathbf{B}) + \varepsilon_0 c^2 \nabla \cdot \frac{\partial \mathbf{A}}{\partial \mathbf{t}}\,, \tag{5}$$

since the divergence of a curl is null. Let us use the following identity:

$$\mathbf{a} \cdot (\nabla \times \mathbf{b}) = \nabla \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{b} \cdot (\nabla \times \mathbf{a})\,. \tag{6}$$

Equation (5) becomes, with $\mathbf{a} = \nabla\phi$, $\mathbf{b} = \mathbf{B}$, and taking into account that the rotational of a gradient is null,

$$\nabla \cdot \mathbf{S}' = \varepsilon_0 c^2 \nabla(\mathbf{B} \times \nabla\phi) + \varepsilon_0 c^2 \nabla \cdot \frac{\partial \mathbf{A}}{\partial \mathbf{t}}\,.$$

Injecting eq. (3) into this last equation leads to

$$\nabla \cdot \mathbf{S}' = -\varepsilon_0 c^2 \nabla(\mathbf{B} \times \mathbf{E}) = \varepsilon_0 c^2 \nabla(\mathbf{E} \times \mathbf{B}) = \nabla\mathbf{S}\,.$$

Quod erat demonstrandum.