Consider that the wave function of a dimensionless harmonic oscillator, whose Hamiltonian is $\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2$, is given at time $t = 0$ by

$$\psi(x,0) = \frac{1}{\sqrt{8\pi}}\phi_0(x) + \frac{1}{\sqrt{18\pi}}\phi_2(x) = \frac{1}{\sqrt{8\pi}}\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{18\pi}}\left(1-2x^2\right)\exp\left(-\frac{x^2}{2}\right)$$

The oscillator's wave function at a time t is given by

$$\Psi(x,t) = \frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp\left(-\frac{iE_2 t}{\hbar}\right)$$

Where E_0 and E_2 are given by $\frac{1}{2}$ and $\frac{5}{2}$ respectively. This was derived in a previous exercise. For ϕ_0 and ϕ_2, I will leave out the (x) for compactness.

Calculating Probability Density

Using $\sqrt{8\pi}\sqrt{18\pi} = 12\pi$:

$$\begin{aligned}\rho(x,t) = \Psi^*(x,t)\Psi(x,t) &= \left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(\frac{iE_0 t}{\hbar}\right) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp\left(\frac{iE_2 t}{\hbar}\right)\right)\left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1}{\sqrt{18\pi}}\phi_2 \exp\left(-\frac{iE_2 t}{\hbar}\right)\right) \\ &= \frac{1}{8\pi}\phi_0^2 + \frac{1}{18\pi}\phi_2^2 + \frac{1}{12\pi}\phi_0\phi_2\left(\exp\left(\frac{i(E_0-E_2)t}{\hbar}\right) + \exp\left(\frac{i(E_2-E_0)t}{\hbar}\right)\right)\end{aligned}$$

Using $\exp(i\xi) + \exp(-i\xi) = 2\cos(\xi)$ where $\xi = \frac{(E_2-E_0)t}{\hbar}$ yields:

$$\boxed{\rho(x,t) = \frac{1}{8\pi}\phi_0^2 + \frac{1}{18\pi}\phi_2^2 + \frac{1}{6\pi}\phi_0\phi_2 \cos\left(\frac{(E_2-E_0)t}{\hbar}\right)}$$

Calculating Current Density

I will first write $\Psi(x,t)$ and $\Psi^*(x,t)$ only in terms of ϕ_0.

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0 \exp\left(-\frac{iE_2 t}{\hbar}\right) \\ \Psi^*(x,t) &= \frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(\frac{iE_0 t}{\hbar}\right) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0 \exp\left(\frac{iE_2 t}{\hbar}\right)\end{aligned}$$

For reference, the derivative of ϕ_0 is $-x\phi_0$:

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(-\frac{x^2}{2}\right)\right) = \exp\left(-\frac{x^2}{2}\right)\left(-\frac{2x}{2}\right) = -x\exp\left(-\frac{x^2}{2}\right) = -x\phi_0$$

Then the derivatives of $\Psi(x,t)$ and $\Psi^*(x,t)$ are:

$$\frac{\partial\Psi(x,t)}{\partial x} = \frac{1}{\sqrt{8\pi}}\frac{\mathrm{d}\phi_0}{\mathrm{d}x}\exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1}{\sqrt{18\pi}}\exp\left(-\frac{iE_2 t}{\hbar}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left((1-2x^2)\phi_0\right)$$

$$= \frac{-x}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1}{\sqrt{18\pi}}\exp\left(-\frac{iE_2 t}{\hbar}\right)\left((1-2x^2)(-x\phi_0) + \phi_0(-4x)\right) = \frac{-x}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0 \exp\left(-\frac{iE_2 t}{\hbar}\right)$$

$$\frac{\partial\Psi^*(x,t)}{\partial x} = \frac{-x}{\sqrt{8\pi}}\phi_0 \exp\left(\frac{iE_0 t}{\hbar}\right) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0 \exp\left(\frac{iE_2 t}{\hbar}\right)$$

Then, it can be said:

$$\Psi(x,t)\frac{\partial\Psi^*(x,t)}{\partial x} = \left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0 \exp\left(-\frac{iE_2 t}{\hbar}\right)\right)\left(\frac{-x}{\sqrt{8\pi}}\phi_0 \exp\left(\frac{iE_0 t}{\hbar}\right) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0 \exp\left(\frac{iE_2 t}{\hbar}\right)\right)$$

$$= \frac{-x}{8\pi}\phi_0^2 + \frac{(1-2x^2)(2x^3-5x)}{18\pi}\phi_0^2 + \frac{2x^3-5x}{12\pi}\phi_0^2 \exp\left(\frac{i(E_2-E_0)t}{\hbar}\right) + \frac{2x^3-x}{12\pi}\phi_0^2 \exp\left(\frac{i(E_0-E_2)t}{\hbar}\right)$$

And:

$$\Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} = \left(\frac{1}{\sqrt{8\pi}}\phi_0 \exp\left(\frac{iE_0 t}{\hbar}\right) + \frac{1-2x^2}{\sqrt{18\pi}}\phi_0 \exp\left(\frac{iE_2 t}{\hbar}\right)\right)\left(\frac{-x}{\sqrt{8\pi}}\phi_0 \exp\left(-\frac{iE_0 t}{\hbar}\right) + \frac{2x^3-5x}{\sqrt{18\pi}}\phi_0 \exp\left(-\frac{iE_2 t}{\hbar}\right)\right)$$

$$= \frac{-x}{8\pi}\phi_0^2 + \frac{(1-2x^2)(2x^3-5x)}{18\pi}\phi_0^2 + \frac{2x^3-5x}{12\pi}\phi_0^2 \exp\left(\frac{i(E_0-E_2)t}{\hbar}\right) + \frac{2x^3-x}{12\pi}\phi_0^2 \exp\left(\frac{i(E_2-E_0)t}{\hbar}\right)$$

When taking $\Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x}$, the first two terms (without exponentials) cancel out:

$$\Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} =$$

$$\frac{2x^3-5x}{12\pi}\phi_0^2\left(\exp\left(\frac{i(E_2-E_0)t}{\hbar}\right) - \exp\left(\frac{i(E_0-E_2)t}{\hbar}\right)\right) + \frac{2x^3-x}{12\pi}\phi_0^2\left(\exp\left(\frac{i(E_0-E_2)t}{\hbar}\right) - \exp\left(\frac{i(E_2-E_0)t}{\hbar}\right)\right)$$

If one defines $\xi = \frac{(E_2-E_0)t}{\hbar}$, then we have:

$$\Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} = \frac{2x^3-5x}{12\pi}\phi_0^2(\exp(i\xi) - \exp(-i\xi)) + \frac{2x^3-x}{12\pi}\phi_0^2(\exp(-i\xi) - \exp(i\xi))$$

$$= \frac{2x^3-5x}{12\pi}\phi_0^2(\exp(i\xi) - \exp(-i\xi)) + \frac{x-2x^3}{12\pi}\phi_0^2(\exp(i\xi) - \exp(-i\xi))$$

Using $\exp(i\xi) - \exp(-i\xi) = 2i\sin(\xi)$:

$$\Psi(x,t)\frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial x} = \left(\frac{2x^3-5x}{12\pi} + \frac{x-2x^3}{12\pi}\right)\phi_0^2(2i)\sin\left(\frac{(E_2-E_0)t}{\hbar}\right)$$

$$= 2i\frac{-4x}{12\pi}\phi_0^2 \sin\left(\frac{(E_2-E_0)t}{\hbar}\right)$$

The term that was just calculated needs to be multiplied by $\frac{i\hbar}{2m}$ to yield the current density J. Thus:

$$J(x,t) = \frac{i\hbar}{2m}2i\frac{-4x}{12\pi}\phi_0^2 \sin\left(\frac{(E_2-E_0)t}{\hbar}\right) = \boxed{\frac{\hbar x}{3\pi m}\phi_0^2 \sin\left(\frac{(E_2-E_0)t}{\hbar}\right)}$$

Continuity Equation

In this case, the continuity equation states:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

The partial time derivative of probability density is as follows. (Note that the first two terms of the probability density on page 1 do not depend on t and thus are reduced to zero. I will also express $-\phi_2$ as $(2x^2-1)\phi_0$).

$$\frac{\partial \rho}{\partial t} = \frac{1}{6\pi}\phi_0\phi_2(-)\sin\left(\frac{(E_2-E_0)t}{\hbar}\right)\left(\frac{E_2-E_0}{\hbar}\right) = \frac{2x^2-1}{6\pi}\frac{E_2-E_0}{\hbar}\phi_0^2 \sin\left(\frac{(E_2-E_0)t}{\hbar}\right)$$

The partial spatial derivative of the probability current requires the product rule (as ϕ_0 depends on x):

$$\frac{\partial J}{\partial x} = \frac{\hbar}{3\pi m}\sin\left(\frac{(E_2-E_0)t}{\hbar}\right)\frac{\mathrm{d}}{\mathrm{d}x}(x\phi_0^2)$$

Using $\phi_0^2 = \exp\left(-\frac{x^2}{2}\right)^2 = \exp(-x^2)$, the derivative component is:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\phi_0^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x\exp(-x^2)) = x\exp(-x^2)(-2x) + \exp(-x^2)(1) = (-2x^2+1)\phi_0^2 = -(2x^2-1)\phi_0^2$$

Hence:

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = \frac{2x^2 - 1}{6\pi} \frac{E_2 - E_0}{\hbar} \phi_0^2 \sin\left(\frac{(E_2 - E_0)\,t}{\hbar}\right) - \frac{2x^2 - 1}{6\pi} \frac{2\hbar}{m} \phi_0^2 \sin\left(\frac{(E_2 - E_0)\,t}{\hbar}\right)$$
$$= \frac{2x^2 - 1}{6\pi} \phi_0^2 \sin\left(\frac{(E_2 - E_0)\,t}{\hbar}\right)\left(\frac{E_2 - E_0}{\hbar} - \frac{2\hbar}{m}\right) \neq 0$$

Even substituting in $E_2 - E_0 = \frac{5}{2} - \frac{1}{2} = 2$, this still doesn't satisfy the continuity equation.