# Homework 6

### Problem 1: Spaceship near a black hole

The pilot turns off the engine, and the spaceship travels in a circular path around a black hole of mass M. The geometry outside is the Schwarzschild geometry. The Schwarzschild radius of the spaceship's path is $7M$.

a) What is the period of the path as measured by an observer at infinity? (remember $T = \frac{2\pi}{\Omega}$ and $\Omega = \frac{d\phi}{dt}$).

b) What is the period of the path as measured by a clock in the spaceship?

### Problem 2: Rocket trip

Observer $O_1$ is in a rocket that is moving in a circular trajectory, in a space-time with the Schwarzschild metric. The trajectory is characterized by:

$$r > 2M,\ \theta = \frac{\pi}{2},\ \frac{d\phi}{dt} = \omega \neq 0$$

where $\omega$ is the angular velocity. The non vanishing Christoffel signs are:

$$\begin{aligned} \Gamma^r_{tt} &= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \\ \Gamma^r_{\phi\phi}(\theta = \pi/2) &= -r\left(1 - \frac{2M}{r}\right) \\ \Gamma^r_{rr} &= -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \end{aligned}$$

a) For what values of r does the observer not need the rocket to stay on the trajectory (in short, if he were out in space he might die for lack of oxygen but would keep on going in the same path).

b) What are the possible values of $\omega$ if he can use the rocket's engine?

c) For which values of r is the upper limit of $|\omega|$ maximal, and what is the upper limit in that case?

d) Another observer, $O_2$ is at

$$r = R,\ \theta = \frac{\pi}{2},\ \phi = \phi_2$$

He sends out radiation with wave length $\lambda$. This radiation reaches a third observer, $O_3$, at

$$r = L > R,\ \theta = \frac{\pi}{2},\ \phi = \phi_3$$

with wave length $\lambda'$. Calculate the redshift, $\frac{\lambda'}{\lambda}$.

**Problem 3 : Asteroid damage**

A spaceship orbits around a black hole of mass M, in a circular orbit of radius R. A little asteroid hits it, and causes it to move radially away from its orbit.

a) For a stable orbit, the perturbation has the form $\Delta r \propto cos(\omega\tau)$. Calculate $\omega$.

b) What is the condition that the new orbit will be closed?

c) For an unstable orbit, the perturbation grow with time as

$$\Delta r \propto e^{\omega\tau}$$

Calculate $\omega$.

d) Explain the results of questions a,b,a for r=6GM.

**Problem 3 : Light near a star**

A radiating source of light is at distance R from a star with mass M (R>2MG). In class we derived the angle of light that escapes to infinity,

$$tan\alpha \quad = \quad \left(\sqrt{\frac{r^2}{b^2\left(1-\frac{2M}{r}\right)}} - 1\right)^{-1}$$

Find $\alpha_{crit}$, the critical angle where light will fall into the star (and past the potential barrier).

## Problem 5: Radially free falling.

Find the differential equation for radially free falling particle from infinity with $l = 0$ and $e = 1$ ($\epsilon = 0$), express the equation in terms of proper time $r(\tau)$ and Schwarzschild time $r(t)$, solve the integral and find its dependence explicitly.