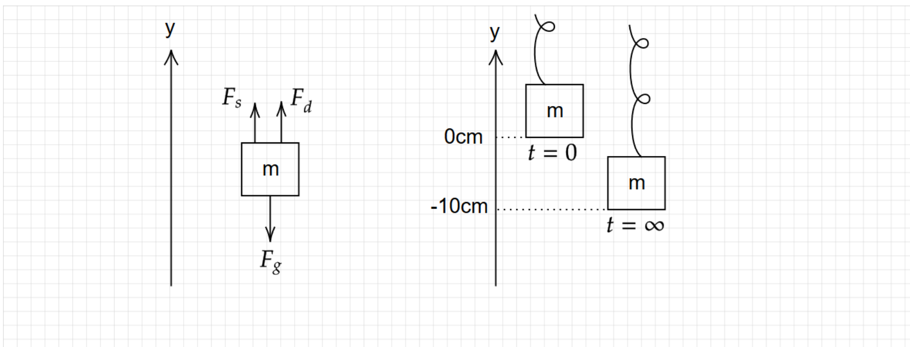
1 Exercise 2.3

A critically mechanical system consisting of a pan hanging from a spring with a damping. What is the value of damping force r if a mass extends the spring by 10cm without overshoot. The mass is 5kg. ($g = 9.81$).



A big thanks to the users of Physics Forum. The solution is taken from this thread: `https://www.physicsforums.com/threads/find-the-resistive-constant-in-a-critically-damped-system.954219/`

2 Solution to the problem

The three forces that acts on the pan are:

- force of the spring: $F_{\text{spring}} = sy$;
- force of the damping: $F_{\text{damping}} = r\dot{y}$;
- force of gravity: $F_{\text{gravity}} = mg$.

Due to Newton's second law, we have

$$m\ddot{y} = F_{\text{spring}} + F_{\text{damping}} - F_{\text{gravity}} = sy + r\dot{y} - mg$$

We can rewrite this to

$$m\ddot{y} + r\dot{y} + sy = mg$$

The system is critically damped:

$$r^2 - 4ms = 0 \qquad \Rightarrow \qquad s = \frac{r^2}{4m}$$

Then we analyze the situation at $t = \infty$: the pan is in rest position, the velocity is zero so the damping force is zero too. The pan is in equilibrium position so the forces F_{gravity} and F_{spring} are one the opposite of the other. So

$$F_{\text{spring}} = F_{\text{gravity}}$$

$$s\Delta y = mg$$

$$s = \frac{mg}{\Delta y}$$

From the previous formula, we get:

$$r = \sqrt{\frac{s}{4m}} = \sqrt{\frac{4m^2 g}{\Delta y}} = 2m\sqrt{\frac{g}{\Delta y}}$$

Finally we obtain:

$$\ddot{y} + 2\sqrt{\frac{g}{\Delta y}}\dot{y} + \frac{g}{\Delta y}y = g$$

$$\ddot{y} + \quad 2\omega_0\,\dot{y} \quad + \omega_0^2\,y = \omega_0^2\,\Delta y$$

3 Solution to the equation of motion

This linear second order ODE is critically damped so the solution has the form:

$$y(t) = e^{-\omega_0 t}(A + Bt)$$

Because of the constant term on the right, I tried something different:

$$y(t) = e^{-\omega_0 t}(A + Bt) + \Delta y$$

which derivatives are

$$\begin{aligned}\dot{y}(t) &= Be^{-\omega_0 t} - \omega_0(A + Bt)e^{-\omega_0 t} \\ \ddot{y}(t) &= \omega_0^2(A + Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t}\end{aligned}$$

which we can substitute into the second order equation to obtain

$$\begin{aligned}&\left[\omega_0^2(A + Bt)e^{-\omega_0 t} - 2B\omega_0 e^{-\omega_0 t}\right] \\ +\,2\omega_0 &\left[\quad Be^{-\omega_0 t} - \omega_0(A + Bt)e^{-\omega_0 t} \quad\right] \\ +\,\omega_0^2 &\left[\qquad (A + Bt)e^{-\omega_0 t} + \Delta y \qquad\right] = \omega_0^2\,\Delta y\end{aligned}$$

which happens to be $0 = 0$ which is what we expect from a linear *homogeneous* second order differential equation.

3.0.1 Find the coefficients

Initially the pan is in rest position so $\dot{y}(0) = 0$. Because of the coordinate system, the initial position is $y(0) = 0$. We substitute into the derivatives in order to find A, B:

$$\begin{aligned}y(0) &= 0 \\ e^0(A + 0) + \Delta y &= 0 \\ A &= -\Delta y\end{aligned}$$

$$\begin{aligned}\dot{y}(0) &= 0 \\ Be^0 - \omega_0(A + 0)e^0 &= 0 \\ B &= -\omega_0\,\Delta y\end{aligned}$$

3.0.2 The actual solution

$$y(t) = -\Delta y\,(\omega_0 t + 1)e^{-\omega_0 t} + \Delta y$$

4 Solution generated by Mathematica

Follows the output from Mathematica software.

```
In[1]:= system = y''[t] + 2 * omega0 * y'[t] + omega0^2 * y[t] == omega0^2 * DY
Out[1]= omega0^2 y[t] + 2 omega0 y′[t] + y″[t] == DY omega0^2

In[2]:= solution = Expand[DSolve[{system && y[0] == 0 && y'[0] == 0}, y[t], t]]
Out[2]= {{y[t] → DY - DY e^(-omega0 t) - DY e^(-omega0 t) omega0 t}}

In[5]:= solution = Collect[solution, Exp[-omega0 * t]]
Out[5]= {{y[t] → DY + e^(-omega0 t) (-DY - DY omega0 t)}}

In[21]:= Plot[
          Evaluate[{y[t] /. solution /. omega0 → Sqrt[9.81 / DY] /. DY → 0.1}],
          {t, 0, 1.5},
          PlotRange → All]
```

Out[21]=