Using the frequency-domain approach, one expresses the excitation or input signal in terms of its Fourier series or Fourier integral, and the system is expressed in terms of its gain or transfer function $H(f)$ (defined as the ratio of output to input when both vary as $e^{j\omega t}$). Each component in the input of the form $e^{j\omega t}$ thus is modified, in passing through the linear system, by being multiplied by the factor $H(f)$. If the input

H(f)

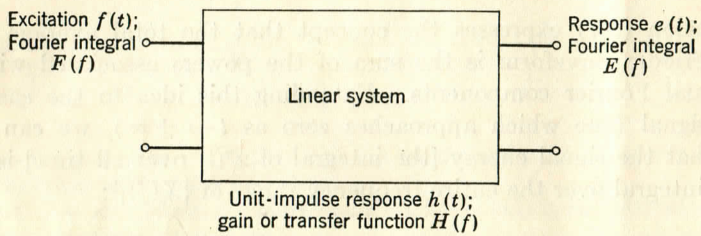

Fig. 2-9. Linear-system representation relevant to the discussion of Parseval's theorem.

Fourier transform is $F(f)$, the transform of the output is $F(f)H(f)$. Thus the output signal (response) is

$$\boxed{e(t) = \int_{-\infty}^{\infty} F(f)H(f)e^{j\omega t}\,df} \tag{2-26}$$

For example, a unit-impulse excitation $f(t) = \delta(t)$ has the Fourier transform $F(f) = 1$. Thus the response will have the transform $H(f) \cdot 1$, and

$$e(t) = h(t) = \int_{-\infty}^{\infty} H(f)e^{j\omega t}\,df \tag{2-27}$$

We shall use the symbol $h(t)$ to designate the unit-impulse response of a linear system. The gain function $H(f)$ is seen to be the Fourier transform of the unit-impulse response $h(t)$. yes, since $F(f)=1$ here

In approaching the same example, using the time domain, one may assume the input to be made up of impulses of width $d\tau$ and height $f(\tau)$ (Fig. 2-10). In the limit as $d\tau$ becomes vanishingly small, each impulse produces an output signal which is, at time t, equal to $f(\tau)\,d\tau\,h(t-\tau)$. The total output at time t is obtained by summing the output components due to all the individual input impulses up to time t:

$$e(t) = \int_{-\infty}^{t(\text{or } \infty)} f(\tau)h(t-\tau)\,d\tau \tag{2-28}$$

[Here the upper limit can be ∞ as well as t, since for a passive system $h(t-\tau)$ is zero for all τ greater than t.] This is termed the *superposition integral* or *convolution integral*.

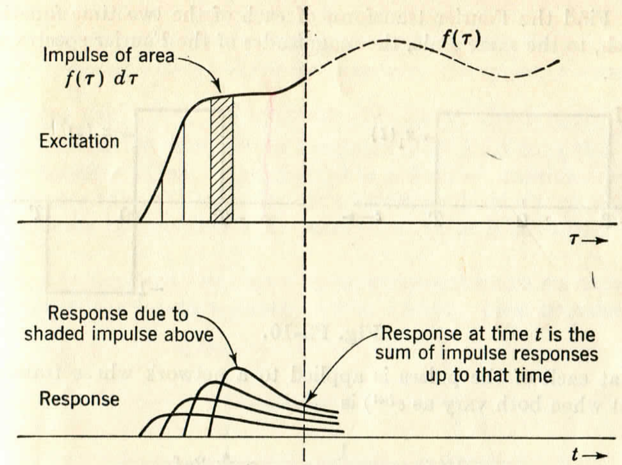

Fig. 2-10. Derivation of the superposition integral.

The equivalence of these two approaches can be shown by taking the Fourier transform of both sides of the superposition integral:

$$\begin{aligned} E(f) &= \int_{-\infty}^{\infty} e(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)h(t-\tau)e^{-j\omega t}\,d\tau\,dt \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega(t-\tau)}\,dt\right]d\tau \\ &= F(f)H(f) \end{aligned} \tag{2-29}$$

Thus we have the convolution-integral theorem

$$e(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} F(f)H(f)e^{j\omega t}\,df \tag{2-30}$$

Setting $t = 0$ gives the relation known as Parseval's theorem:

$$\int_{-\infty}^{\infty} f(\tau)h(-\tau)\,d\tau = \int_{-\infty}^{\infty} F(f)H(f)\,df \tag{2-31}$$

This is put in more usual form by inserting the function $g(t) \triangleq h(-t)$ with the Fourier transform $G(f)$, yielding

$$\int_{-\infty}^{\infty} f(t)g(t)\,dt = \int_{-\infty}^{\infty} F(f)G^*(f)\,df \tag{2-32}$$

Although this relationship was obtained here in connection with a particular physical example, it is a general mathematical identity, and the functions $f(t)$ and $g(t)$ could be any real functions that have Fourier transforms.[1]

[1] If $f(t)$ and $g(t)$ are general functions which may be complex, $g(t)$ should be replaced by $g^*(t)$ in Eq. (2-32).