# $f(f(x)) = -x$

rash

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## 1 Generate an $f$

Partition $\mathcal{R}\backslash\{0\}$ into sets $A$, $B$ s.t. $A$ is closed over negation and $A \sim B$.

**Lemma 1.** $B$ is closed over negation.

Define an odd bijection $g : A \to B$.

**Lemma 2.** $g^{-1}$ is an odd bijection from $B \to A$

Define $h : B \to A$ s.t. $h(b) = -g^{-1}(b)$. h is an odd bijection.

$$f(x) = \begin{cases} 0 & | \quad x = 0 \\ g(x) & | \quad x \in A \\ h(x) & | \quad x \in B \end{cases}$$

**Lemma 3.** For $x \in \mathcal{R}$, $f(f(x)) = -x$

The piecewise definition captures the tricks we use, e.g. switching between evens and odds for $\mathcal{Z}$, doing the same for denominators of non-integer rationals, and using $\frac{1}{x}$ and $-\frac{1}{x}$ to map $\{x \in \mathcal{R} || x| < 1\}$ to $\{x \in \mathcal{R} || x| > 1\}$, etc.

## 2 Properties of $f$

We found some conditions of a function $f$ that are sufficient to show $f(f(x)) = -x$. Are these characteristics true of all such functions $f$ over $\mathcal{R}$ satisfying that equality?

Suppose we have a function $f : \mathcal{R} \to \mathcal{R} : f(f(x)) = -x$.

### 2.1 $f$ odd

Take $f(-x) = h$
$f(f(-x)) = f(h) = x$
$f(f(h)) = f(x) = -h$

Thus $f(-x) = h = -(-h) = -f(x)$

$f$ odd.

### 2.2 $f$ surjective

Take any $x \in \mathcal{R}$. f is defined on all $\mathcal{R}$.

Thus we have $f(x) \in \mathcal{R}$,
$f(f(x)) = -x \in \mathcal{R}$,
$f(f(f(x))) = -f(x) \in \mathcal{R}$, and
$f^4(x) = f(f(-x)) = x$.

Thus $\forall x \in \mathcal{R}$, $\exists y = f^3(x)$ with $y \in \mathcal{R}$ and $f(y) = x$.

$f$ surjective.

### 2.3 $f$ injective.

Say $f(x) = f(y)$.

Then $f(f(x)) = f(f(y)) \Rightarrow -x = -y \Rightarrow x = y$.

$f$ injective.

$f$ a bijection.

The mapping $f \subset \mathcal{R}^2$ can be partitioned in a special way.

Consider all non-zero cycles in $\mathcal{R}$. Each cycle yields a set of four elements.

Lemma 4. Each nonzero cycle contains four unique elements of $\mathcal{R}$.

Therefore each nonzero cycle has a unique maximum and minimum.

Lemma 5. If the max of a cycle is $a$, the min is $-a$

Define $A = \{a \in \mathcal{R} | a$ is the max or min of a nonzero cycle$\}$.

Define $B = f(A)$.

## 2.4 $A \cap B = \emptyset$

Say $x \in A \cap B$.

$x \in B \Rightarrow$ for some $a \in A, x = f(a)$. This defines a cycle $a \to x \to -a \to -x$.

$a \in A \Rightarrow a$ or $-a$ is the minimum.
$x \in A \Rightarrow x$ or $-x$ is the minimum.

This contradicts the uniqueness lemma; $A \cap B = \emptyset$.

## 2.5 $A \cup B \cup \{0\} = \mathcal{R}$

Say $x \in \mathcal{R}, x \neq 0, x \notin A$, and $x \notin B$.

$x \in \mathcal{R} \backslash \{0\} \Rightarrow f$ defines a cycle of four unique elements for $x : x, f(x), -x, -f(x)$. The set of four unique reals has a minimum and maximum value which are elements of $A$. The other two elements are the images of the maximum and minimum over $f$; they are elements of $B$. Thus $x \in A \cup B$.

Because $A \cup B \cup \{0\}$ is a partition on $\mathcal{R}$, the domain of $f$, $f$ can be piecewise defined:

$$f(x) = \begin{cases} 0 & | \quad x = 0 \\ f_A(x) & | \quad x \in A \\ f_B(x) & | \quad x \in B \end{cases}$$

## 2.6 $f(B) = A$

### 2.6.1 $f(B) \subset A$

Take $b \in B$. $f(b) = r \in \mathcal{R}$. $0 \notin B$, so $b \neq 0$ and cannot generate a zero cycle. Thus $f(b) \in A$ or $f(b) \in B$.

$f(b) = r \in B \Rightarrow \exists a \in A$ s.t. $f(a) = r$, because $B = f(A)$.

$f$ an injection means $f(b) = r = f(a) \Rightarrow b = a$, contradicting $A \cap B = \emptyset$. Thus $r \notin B \Rightarrow r \in A$.

### 2.6.2 $A \subset f(B)$

Say $a \in A$. $f$ a bijection on $\mathcal{R} \to \mathcal{R} \Rightarrow \exists r \in \mathcal{R}$ with $f(r) = a$. $r$ cannot be zero ($a \in A \Rightarrow a \neq 0$). $r \in A \Rightarrow r$ a max or min and $a \in A \Rightarrow a$ a max or min $\Rightarrow r = a$ or $r = -a$. $r = a$ contradicts uniqueness. $r = -a$ generates cycle $r \to a \to -r \to -a$ or, equivalently, $r \to a \to a \to r$, also contradicting uniqueness. $r \in B$.

Thus $\forall a \in A \ \exists r \in B$ with $f(r) = a$.

$\boxed{f(B) = A}$.

## 2.7 $A, B$ closed over negation

$A$ is closed over negation by its definition.

Take $b \in B$. Say $-b \in A$. Then its negation, $b$, is also in $A$, contradicting $A \cap B = \emptyset$. Thus $b \in B \Rightarrow -b \notin A \Rightarrow -b \in B$.

### 2.8 $A \sim B$

$f(A) = B$.

$f$ a bijection.

Take $g(x) = f_A(x), h(x) = f_B(x)$.

### 2.9 $h$ and $g$ are odd bijections over their restrictions

They inherit such properties from f.

### 2.10 $h(b) = -g^{-1}(b)$

For $b \in B, f(f(b)) = -b$.

$f(b) = f_B(b) = h(b)$.

As $f(b) \in A, f(f(b)) = f(h(b)) = f_A(h(b)) = g(h(b))$.

$f(f(b)) = g(h(b)) = -b \Rightarrow h(b) = g^{-1}(-b) \Rightarrow h(b) = -g^{-1}(b)$. Recall that $g$ is an odd bijection on $A \to B$, so $g^{-1}$ exists as an odd bijection on $B \to A$.

Therefore we have characterized all of $f$. Maybe.