1 Rewriting the hamiltonian

1.1 The hamiltonian in question

$$\begin{aligned}\mathcal{H} = & -t_e \sum_{r,r',\sigma} c^\dagger_{r,\sigma} c_{r',\sigma} - \mu \sum_{r,\sigma} c^\dagger_{r,\sigma} c_{r,\sigma} + i\alpha \sum_{r,\delta,\sigma,\sigma'} c^\dagger_{r,\sigma} (\boldsymbol{\delta} \times \boldsymbol{\sigma})^z_{\sigma,\sigma'} c_{r+\delta,\sigma'} \\ & + \Delta \sum_r \left(c^\dagger_{r,\uparrow} c^\dagger_{r,\downarrow} + c_{r,\downarrow} c_{r,\uparrow} \right) + \sum_{R,\sigma,\sigma'} c^\dagger_{R,\sigma} \left[J\mathbf{S}_R \cdot \boldsymbol{\sigma} \right]_{\sigma,\sigma'} c_{R,\sigma'}\end{aligned} \tag{1}$$

1.2 Fourier transform

$$\begin{aligned}\tilde{\mathcal{H}} = & -t_e \sum_{k,\sigma} \varepsilon_k c^\dagger_{k,\sigma} c_{k,\sigma} - \mu \sum_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} + i\alpha \sum_{k,\sigma,\sigma'} \eta_{k,\sigma,\sigma'} c^\dagger_{k,\sigma} c_{k,\sigma'} \\ & + \Delta \sum_k \left(c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + c_{k,\downarrow} c_{-k,\uparrow} \right) + \sum_{k,\sigma,\sigma'} \omega_{\sigma,\sigma'} c^\dagger_{k,\sigma} c_{k,\sigma'}\end{aligned} \tag{2}$$

where

$$\varepsilon_k = \sum_\delta e^{ik\delta} \tag{3}$$

$$\eta_{k,\sigma,\sigma'} = \sum_\delta e^{ik\delta} (\boldsymbol{\delta} \times \boldsymbol{\sigma})^z_{\sigma,\sigma'} \tag{4}$$

$$\omega_{\sigma,\sigma'} = \left[J\mathbf{S}_R \cdot \boldsymbol{\sigma} \right]_{\sigma,\sigma'} \tag{5}$$

1.3 Regroup terms in common sums

$$\begin{aligned}\tilde{\mathcal{H}} = & \sum_{k,\sigma} \varepsilon'_k c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_{k,\sigma,\sigma'} \omega'_{k,\sigma,\sigma'} c^\dagger_{k,\sigma} c_{k,\sigma'} \\ & + \Delta \sum_k \left(c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + c_{k,\downarrow} c_{-k,\uparrow} \right)\end{aligned} \tag{6}$$

where

$$\varepsilon'_k = (-t_e \varepsilon_k - \mu) \tag{7}$$

$$\omega'_{k,\sigma,\sigma'} = (i\alpha \eta_{k,\sigma,\sigma'} + \omega_{\sigma,\sigma'}) \tag{8}$$

1.4 Expand on σ

$$\tilde{\mathcal{H}} = \sum_k \varepsilon'_k \left(c^\dagger_{k,\uparrow} c_{k,\uparrow} + c^\dagger_{k,\downarrow} c_{k,\downarrow} \right) + \omega'_{k,\uparrow,\uparrow} c^\dagger_{k,\uparrow} c_{k,\uparrow} + \omega'_{k,\uparrow,\downarrow} c^\dagger_{k,\uparrow} c_{k,\downarrow} + \omega'_{k,\downarrow,\uparrow} c^\dagger_{k,\downarrow} c_{k,\uparrow} + \omega'_{k,\downarrow,\downarrow} c^\dagger_{k,\downarrow} c_{k,\downarrow}$$
$$+ \Delta c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + \Delta c_{k,\downarrow} c_{-k,\uparrow} \tag{9}$$

Next we regroup terms again

$$\tilde{\mathcal{H}} = \sum_k (\varepsilon'_k + \omega'_{k,\uparrow,\uparrow})\, c^\dagger_{k,\uparrow} c_{k,\uparrow} + (\varepsilon'_k + \omega'_{k,\downarrow,\downarrow})\, c^\dagger_{k,\downarrow} c_{k,\downarrow} + \omega'_{k,\uparrow,\downarrow}\, c^\dagger_{k,\uparrow} c_{k,\downarrow} + \omega'_{k,\downarrow,\uparrow}\, c^\dagger_{k,\downarrow} c_{k,\uparrow}$$
$$+ \Delta c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} + \Delta c_{k,\downarrow} c_{-k,\uparrow} \tag{10}$$

1.5 Expand on $-k$ and k

We also anticommute the operators so that they are consisten with the operators we are going to use, namely:

$$\psi^\dagger = (c^\dagger_{k,\uparrow}, c^\dagger_{k,\downarrow}, c_{-k,\downarrow}, c_{-k,\uparrow}) \tag{11}$$

$$\psi = (c_{k,\uparrow}, c_{k,\downarrow}, c^\dagger_{-k,\downarrow}, c^\dagger_{-k,\uparrow}) \tag{12}$$

So the Hamiltonian ends up as:

$$\tilde{\mathcal{H}} = \sum_k (\varepsilon'_k + \omega'_{k,\uparrow,\uparrow})\, c^\dagger_{k,\uparrow} c_{k,\uparrow} + (\varepsilon'_k + \omega'_{k,\downarrow,\downarrow})\, c^\dagger_{k,\downarrow} c_{k,\downarrow} + \omega'_{k,\uparrow,\downarrow}\, c^\dagger_{k,\uparrow} c_{k,\downarrow} + \omega'_{k,\downarrow,\uparrow}\, c^\dagger_{k,\downarrow} c_{k,\uparrow}$$
$$- (\varepsilon'_{-k} + \omega'_{-k,\uparrow,\uparrow})\, c_{-k,\uparrow} c^\dagger_{-k,\uparrow} - (\varepsilon'_{-k} + \omega'_{-k,\downarrow,\downarrow})\, c_{-k,\downarrow} c^\dagger_{-k,\downarrow} - \omega'_{-k,\uparrow,\downarrow}\, c_{-k,\downarrow} c^\dagger_{-k,\uparrow} - \omega'_{-k,\downarrow,\uparrow}\, c_{-k,\uparrow} c^\dagger_{-k,\downarrow}$$
$$+ \Delta\, c^\dagger_{k,\uparrow} c^\dagger_{-k,\downarrow} - \Delta\, c^\dagger_{k,\downarrow} c^\dagger_{-k,\uparrow} - \Delta\, c_{k,\downarrow} c_{-k,\uparrow} + \Delta\, c_{k,\uparrow} c_{-k,\downarrow} \tag{13}$$

1.6 Write the matrix

Knowing that

$$\psi^\dagger H \psi = \tilde{\mathcal{H}} \tag{14}$$

We can figure out that:

$$H = \begin{bmatrix} (\varepsilon'_k + \omega'_{k\uparrow\uparrow}) & \omega'_{k\uparrow\downarrow} & \Delta & 0 \\ \omega'_{k\downarrow\uparrow} & (\varepsilon'_k + \omega'_{k\downarrow\downarrow}) & 0 & -\Delta \\ \Delta & 0 & -(\varepsilon'_k + \omega'_{k\downarrow\downarrow}) & -\omega'_{k\downarrow\uparrow} \\ 0 & -\Delta & -\omega'_{k\uparrow\downarrow} & -(\varepsilon'_k + \omega'_{k\uparrow\uparrow}) \end{bmatrix} \tag{15}$$

1.7 Simplifying $\omega'_{k\sigma\sigma'}$

First let's remember that $\omega'_{k,\sigma,\sigma'} = (i\alpha\eta_{k,\sigma,\sigma'} + \omega_{\sigma,\sigma'})$, and that $\eta_{k,\sigma,\sigma'} = \sum_\delta e^{ik\delta}(\boldsymbol{\delta} \times \boldsymbol{\sigma})^z_{\sigma,\sigma'}$ and $\omega_{\sigma,\sigma'} = [J\mathbf{S}_R \cdot \boldsymbol{\sigma}]_{\sigma,\sigma'}$:

$$S_R \cdot \sigma = S^z_R \sigma_z = S^z_R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\omega_{\sigma,\sigma'} = [J\mathbf{S}_R \cdot \boldsymbol{\sigma}]_{\sigma,\sigma'} = \begin{pmatrix} \omega_{\uparrow\uparrow} = JS^z_r & \omega_{\uparrow\downarrow} = 0 \\ \omega_{\downarrow\uparrow} = 0 & \omega_{\downarrow\downarrow} = -JS^z_r \end{pmatrix} = JS^z_r \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}$$

Next, considering nearest neighbors and a square lattice:

$$\delta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\eta_{k,\sigma,\sigma'} = \sum_\delta e^{ik\delta}(\boldsymbol{\delta} \times \boldsymbol{\sigma})^z_{\sigma,\sigma'} = (e^{ik_x} - e^{-ik_x})(\sigma_y)_{\sigma\sigma'} + (-e^{ik_y} + e^{-ik_y})(\sigma_x)_{\sigma\sigma'} = 2i\ sin(k_x)(\sigma_y)_{\sigma\sigma'} - 2i\ sin(k_y)(\sigma_x)_{\sigma\sigma'}$$

With this, we can write $\omega'_{k,\sigma,\sigma'}$ as a matrix too:

$$\omega'_{k,\sigma,\sigma'} = i\alpha\eta_{k,\sigma,\sigma'} + \omega_{\sigma,\sigma'} = \begin{pmatrix} \uparrow\uparrow = JS^z_R & \uparrow\downarrow = 2\alpha(i\ sin(k_x) + sin(k_y)) \\ \downarrow\uparrow = 2\alpha(-i\ sin(k_x) + sin(k_y)) & \downarrow\downarrow = -JS^z_R \end{pmatrix} \tag{17}$$

And we can see that this is already hermitian for all values of k

1.8 Simplifying ε'_k

$$\varepsilon_k = \sum e^{i\delta k} = e^{ik_x} e^{ik_y} + e^{-ik_x} + e^{-ik_y} = 2(cos(k_x) + sin(k_x))$$

$$\varepsilon'_k = -2t_e((cos(k_x) + sin(k_x)) - \mu$$

1.9 Simplifying $\tilde{\mathcal{H}}$

$$\begin{bmatrix} \varepsilon'_k + JS^z_R & 2\alpha(i\ sin(k_x) + sin(k_y)) & \Delta & 0 \\ 2\alpha(-i\ sin(k_x) + sin(k_y)) & \varepsilon'_k - JS^z_R & 0 & -\Delta \\ \Delta & 0 & -\varepsilon'_k + JS^z_R & -2\alpha(i\ sin(k_x) - sin(k_y)) \\ 0 & -\Delta & -2\alpha(-i\ sin(k_x) - sin(k_y)) & -\varepsilon'_k - JS^z_R \end{bmatrix} \tag{18}$$

1.10 Finding eigenvalues

I used Mathematica to find the eigenvalues, and I found they were:

$$\lambda = \pm \Big[4\alpha^2\big(\sin^2 k_x + \sin^2 k_y\big) + \Delta^2 + \varepsilon_k'^2 + (JS_R^z)^2 \\ \pm 2\Big(\varepsilon_k'^2\big(4\alpha^2(\sin^2 k_x + \sin^2 k_y) + (JS_R^z)^2\big) + \Delta^2\big(4\alpha^2\sin^2 k_y + (JS_R^z)^2\big)\Big)^{1/2}\Big]^{1/2} \tag{19}$$

Then, to simplify it, we can use this vector: $\vec{\ell} = (2\alpha sin(k_x), 2\alpha sin(k_y), JS_r^z)$

$$\lambda = \pm\Big[|\vec{\ell}|^2 + \Delta^2 + \varepsilon_k'^2 \pm 2\Big(\varepsilon_k'^2|\vec{\ell}|^2 + \Delta^2\big(\vec{\ell}_y^2 + \vec{\ell}_z^2\big)\Big)^{1/2}\Big]^{1/2} \tag{20}$$