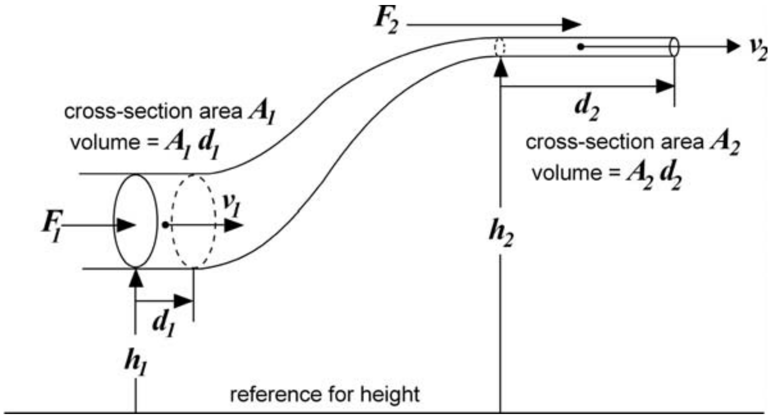
Fluid Flow Energy

Consider a parcel of fluid flowing along an (imaginary) pipe as shown in the diagram.



Select the parcel of fluid to be the system of interest, all else is external to the system. Let an external force F_1 push on the fluid from the left and the fluid push with a force F_2 on the right.
The work done ***on*** the system by the external force F_1 acting through a distance d_1 is F_1d_1.
The work done ***by*** the system on the external by the force F_2 through a distance d_2 is F_2d_2.
The ***net*** work done on the parcel of fluid is thus $F_1d_1 - F_2d_2$ which is energy into the system.

We write the conservation of energy: (frictional forces may generate thermal energy ▬► heat)

$$J_{in\ or\ out} = \Delta E_{kinetic} + \Delta E_{potential} + \Delta E_{\text{internal}}$$

$$F_1\, d_1 - F_2\, d_2 = \left(\frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2\right) + \left(m\, g\, h_2 - m\, g\, h_1\right) + \left(thermal\ energy\right)$$

We now restrict the analysis to an ***incompressible***, viscous fluid; its density (ρ ≡ mass/volume) is a constant everywhere in the fluid. This means as the fluid moves along the pipe through a distance d_1 on the left and simultaneously through a distance d_2 on the right, the amount of mass $\rho A_1 d_1$ around point 1 in the fluid is equal to the mass $\rho A_2 d_2$ around point 2. It is as if the parcel of mass $\rho A_1 d_1 = \rho A_2 d_2$ moved from point 1 to point 2 due to the energy put into the system. We divide the above energy equation on both sides by the volume $A_1 d_1 = A_2 d_2$ of this mass parcel.

$$\frac{F_1\, d_1}{A_1\, d_1} - \frac{F_2\, d_2}{A_2\, d_2} = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + \left(\rho\, g\, h_2 - \rho\, g\, h_1\right) + \left(thermal\ energy\right) \qquad note: m = \rho \cdot volume$$

$$note \quad \frac{F_1}{A_1} = P_1 \ \ the\ pressure\ in\ the\ fluid\ at\ \text{point 1}$$

$$\frac{F_2}{A_2} = P_2 \ \ the\ fluid\ pressure\ at\ \text{point 2}$$

$$P_1 - P_2 = \left(\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2\right) + \left(\rho\, g\, h_2 - \rho\, g\, h_1\right) + \left(thermal\ energy\right)$$

and rearrange to obtain Bernoulli's equation for fluid flow:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho\, g\, h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho\, g\, h_2 + \begin{pmatrix} thermal\ energy \\ per\ unit\ volume \end{pmatrix} \qquad units: energy\ density\ \frac{J}{m^3} = \frac{N}{m^2}$$

In any closed system, energy is neither created nor destroyed, it can only transform.
An ***ideal fluid*** has both constant density and zero internal friction (no thermal energy losses).

Application: Pressure and Power for Fluid Flow

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 + \begin{pmatrix} \text{thermal energy} \\ \text{per unit volume} \end{pmatrix} \qquad units: pressure \ \frac{N}{m^2} = \frac{J}{m^3}$$

In the human body's cardiovascular system, as blood flows farther from the heart (pump source point 1) more thermal energy leaves the system in the form of body heat so the pressure and the average speed of the blood flow decrease. However, for points sufficiently close together energy losses due to heat may be negligible thereby greatly simplifying predictions about fluid flow.

Conservation of mass and flow rate: Rate of mass into a vessel = Rate of mass out of a vessel.

$$mass\ flow\ rate \equiv \frac{mass}{\Delta t} = \frac{\rho \cdot volume}{\Delta t} \qquad units\ \frac{kg}{s}$$

$$volume\ flow\ rate \equiv \frac{volume}{\Delta t} = \frac{Area \cdot \Delta x}{\Delta t} = A \cdot v \qquad units\ \frac{m^3}{s}$$

Fluid's resistance to flow and Poiseuille's Law:

$$volume\ flow\ rate = \frac{pressure\ drop\ across\ vessel}{\text{resistance}\ of\ the\ fluid\ to\ flow}; \quad Q = \frac{\Delta P}{R} \qquad for\ \text{laminar}\ flow:\ R = \frac{8\eta L}{\pi r^4}$$

where: η = measure of fluid's internal friction; L=Length of the vessel; r = radius of the vessel.

¿ Estimate the pressure difference when human blood flows where the diameter of the vessel carrying the fluid (for example the aorta with diameter ≈ 12 mm) decreases by a factor of 5 %? The human heart pumps blood at a volume flow rate of about 5.0 liter/min or 83 cm^3/s.
Blood: density = 1.05 gram/cm^3 = 1050 kg/m^3 ; coefficient of viscosity $\eta = 3.5 \times 10^{-3}$ Pa·s.

A. between two regions, with no internal friction (ideal fluid): (both regions at same height $h_1 = h_2$)

$$v_1 = \frac{volume\ flow\ rate}{cross\ \text{section}\ area} = \frac{83\ cm^3/\sec}{\pi(0.6\,cm)^2} = 73\frac{cm}{s} \qquad and \qquad v_2 = 81\frac{cm}{s}$$

$$P_1 - P_2 = \frac{1}{2}\rho v_1^2 - \frac{1}{2}\rho v_2^2 = \frac{1}{2}\left(1050\frac{kg}{m^3}\right)\left\{\left(0.813\frac{m}{s}\right)^2 - \left(0.734\frac{m}{s}\right)^2\right\} = 64\ Pa$$

B. for a given region, with internal friction (streamline: laminar flow with negligible turbulence):

Resistance of a 5-cm long vessel at radius 6.0 mm $R_1 = \frac{8\,(3.5\,x\,10^{-3}\ Pa\cdot s)(5\,x\,10^{-2}\,m)}{\pi\,(6.0\,x\,10^{-3}\,m)^4} = 3.44\,x\,10^5\ \frac{Pa\cdot s}{m^3}$

Pressure difference at radius 6.0 mm, $\Delta P_1 = Q \cdot R = 83 \times 10^{-6}\ m^3/s \cdot 3.44 \times 10^5 Pa{\cdot}s/m^3 = 29$ Pa

Resistance of a 5-cm long vessel at radius 5.7 mm $R_2 = \frac{8\,(3.5\,x\,10^{-3}\ Pa\cdot s)(5\,x\,10^{-2}\,m)}{\pi\,(5.7\,x\,10^{-3}\,m)^4} = 4.22\,x\,10^5\ \frac{Pa\cdot s}{m^3}$

Pressure difference at radius 5.7 mm, $\Delta P_2 = Q \cdot R = 83 \times 10^{-6}\ m^3/s \cdot 4.22 \times 10^5 Pa{\cdot}s/m^3 = 35$ Pa

C When the pressure in the blood vessels rises, the heart has to pump more to keep the blood circulating. ¿ Estimate the power increase (%) the heart must deliver to maintain the same flow rate to the organs?

Note: pressure increase of 35.1/28.7 = 1.23 (23 %); velocity increase of 81.3/73.4 = 1.11 (11 %).

$$Answer:\ \frac{Power'}{Power} = \frac{F'}{F}\frac{v'}{v} = \frac{P'A'v'}{P\,A\,v} = \frac{P'Q'}{PQ} = 1.23 * 1 = 1.23 = \left(\frac{1}{0.95}\right)^4 \quad or \quad 23\,\% \quad more\ power.$$

A small amount (5%) of arterial occlusion can have a surprisingly large effect on the heart pump!