

5

Gauss' Law

5.1 Introduction

There is an important relation between the vector \mathbf{E} in any electrostatic field and the static charge producing it. It is a consequence of the mathematical form of the electric field strength of a point charge, and is known as Gauss' law. Among other applications, Gauss' law enables a simple evaluation of the electric field in some simple but important cases.

To understand Gauss' law, we first need to understand an important mathematical concept, the *flux of a vector function through a surface*. The word "flux" originates from fluid mechanics and comes from the latin word "fluxus," which means "one that flows."

5.2 The Concept of Flux

Consider a uniform flow of a liquid of velocity \mathbf{v} that is a function of coordinates but not of time. Imagine a net so fine that it does not disturb the flow of the liquid it is placed in. Let the surface of the net be S . We wish to determine the amount of the liquid that passes through the net (i.e., through S) in one second.

We can subdivide the surface S into a large number of small flat surface elements dS , as in Fig. 5.1. Obviously, the total amount of liquid passing through the net is obtained as a sum of the small amounts passing through all of the small elements.

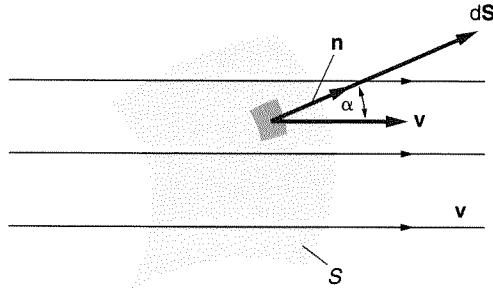


Figure 5.1 A fine net in a flow of liquid can be approximated by a large number of flat surface elements such as dS

Consider a small flat surface element shown in the figure. The vector \mathbf{n} denotes a unit vector normal to the element, and α is the angle between this unit vector and the local velocity \mathbf{v} of the fluid. It is evident that if the velocity \mathbf{v} is tangential to the element, there is no flow of fluid through it. Therefore only the component of the velocity *normal* to the element contributes to the flow of liquid through the element.

In one second, the fluid at that point moves by a distance normal to dS equal to $v \cos \alpha$. The quantity of fluid that passes through dS in one second is therefore $v \cos \alpha dS$. The quantity of fluid that passes through S in one second is a sum of all these infinitely small partial flows. It is therefore an integral (an infinite sum of infinitely small terms):

$$\text{Fluid flow through } S \text{ in one second} = \int_S v \cos \alpha dS. \quad (5.1)$$

The expression under the integral sign has a form of a dot product, but although v is the magnitude of a vector, dS is not. If, however, we *define* a vector surface element $d\mathbf{S}$ as

$$d\mathbf{S} = dS \mathbf{n}, \quad (5.2)$$

Eq. (5.1) can be written in the form

$$\text{Fluid flow through } S \text{ in one second} = \int_S \mathbf{v} \cdot d\mathbf{S}. \quad (5.3)$$

The integral on the right side of this equation is known as the *flux of vector \mathbf{v} through the surface S* .

It is evident that the concept of flux can be used in connection with *any* vector function, not necessarily the velocity (in which case the flux has a clear physical meaning). It is evident as well that the surface S can be a closed surface. In that case, a small circle is added in the middle of the integral sign to indicate that the surface is closed.

The flux of a vector function through a closed surface is a very important concept in the theory of electromagnetic field. It is a convention to adopt the unit vector \mathbf{n} normal to a closed surface *to be directed from the surface outward* (Fig. 5.2).

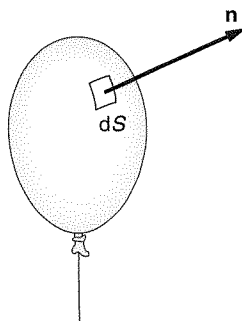


Figure 5.2 The unit vector normal to a closed surface is always adopted to be directed from the surface outward

5.3 Gauss' Law

Gauss' law is a very simple and important consequence of the mathematical form of the expression of the vector \mathbf{E} of a point charge (i.e., of Coulomb's law). It states that the flux of the electric field strength vector through any closed surface in the electrostatic field equals the total charge enclosed by the surface, divided by ϵ_0 :

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{total in } S}}{\epsilon_0} \quad (\text{V} \cdot \text{m}). \quad (5.4)$$

(Gauss' law)

Basically, Gauss' law is a relationship between the sources *inside a closed surface* and the field they produce *over this entire surface*. (For interested readers, the derivation of Gauss' law is given at the end of the chapter.)

Gauss' law in Eq. (5.4) is valid for free space (air, vacuum). We know, however, that elemental charges that are actual sources of the field (electrons, protons, ions) *are* situated in a vacuum. Using this fact, we are able to extend Gauss' law to electrostatic fields in the presence of conducting and dielectric materials.

Example 5.1—Gauss' law applied to point charges. Consider the closed surfaces S_1 , S_2 , and S_3 in Fig. 5.3. The flux of vector \mathbf{E} through S_1 is $(Q_1 + Q_4)/\epsilon_0$, through S_2 is zero, and through S_3 is $(Q_2 + Q_3)/\epsilon_0$.

Questions and problems: Q5.1 to Q5.11, P5.1 to P5.4

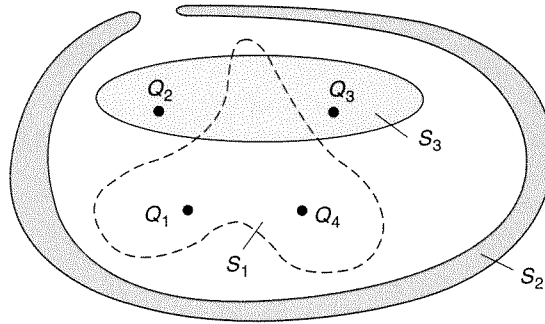


Figure 5.3 Three closed surfaces, S_1 , S_2 and S_3 , in the field of four point charges

5.4 Applications of Gauss' Law

The applications of Gauss' law are numerous. They are basically of two kinds: proofs of some general properties of the electrostatic field, and the evaluation of the vector \mathbf{E} in some special cases with high degree of symmetry of charge distribution.

Example 5.2—Gauss' law applied to a surface of zero field. As an example of the first kind of application, assume that we have a surface S such that \mathbf{E} is zero at all points of S . Gauss' law tells us that in *all* such cases the total enclosed charge must be zero. We will use this conclusion in the analysis of conductors in the next chapter.

Before giving further examples of Gauss' law, we note that it represents a *single* scalar equation. Therefore, in general it is not possible to determine a vector function from it (every vector function is defined by its *three* scalar components). It is possible to use Gauss' law to find \mathbf{E} only if by symmetry we know everything about \mathbf{E} except its magnitude.

Example 5.3—Electric field of an infinite, charged plate. Consider a large, theoretically infinite flat plate uniformly charged with a surface charge density σ (Fig. 5.4a). Due to symmetry, the lines of \mathbf{E} are normal to the plate, and are directed from the plate if $\sigma > 0$ and toward the plate in the other case. What we do not know is the magnitude of \mathbf{E} as a function of the distance x from the plate. We need one scalar equation for that, and Gauss' law can be used. Note that, from symmetry, we know that $E(-x) = E(x)$, and assume that $\sigma > 0$.

Imagine a cylinder of bases S parallel to the plate and of height $2h$, positioned symmetrically with respect to the plate, as in Fig. 5.4a. Let us apply Gauss' law to that closed surface.

On the curved surface, vector \mathbf{E} is parallel to it, i.e., normal to the vector surface element. Therefore, the flux of \mathbf{E} through the curved surface is zero. On the two bases, vector \mathbf{E} is normal to them, i.e., it is parallel to the vector surface element, so the flux of \mathbf{E} through each base is simply $E(x)S$. So we have

$$\oint_{\text{cylinder}} \mathbf{E} \cdot d\mathbf{S} = E(x)S + E(-x)S = 2E(x)S = \frac{\sigma S}{\epsilon_0},$$

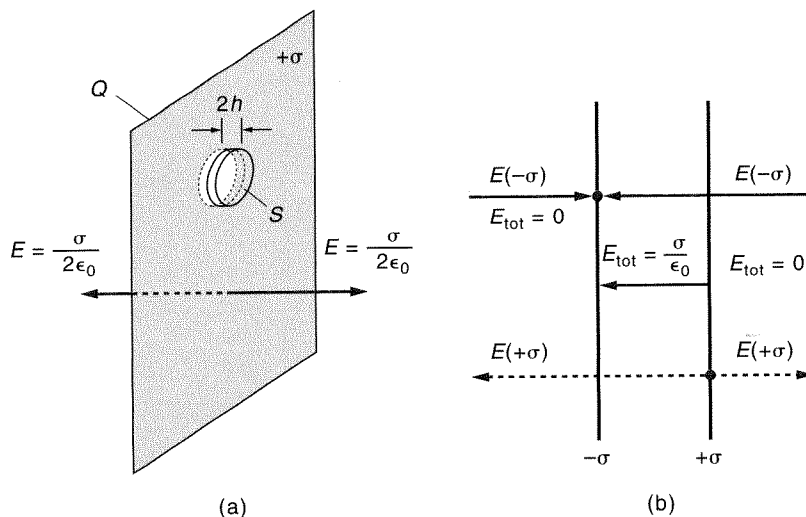


Figure 5.4 (a) A charged plate, and (b) two parallel plates charged with equal surface charges of opposite sign

because the charge enclosed by S is σS . We find that the magnitude E of the electric field strength *does not depend on the distance from the plate*:

$$E = \frac{\sigma}{2\epsilon_0} \quad (\text{V/m}). \quad (5.5)$$

(Electric field strength of uniformly charged plate)

How is it possible that E does not depend on x ? The answer is simple. The plate being theoretically infinite, any finite distance from the plate measured with respect to the plate size is infinitely small; i.e., all points at a finite distance from the plate are equivalent.

Although we cannot have an infinite, uniformly charged plate, the result in Eq. (5.5) is nevertheless of significant importance. If we have a surface charge on a flat (or locally nearly flat) surface of any size and approach it sufficiently close and far from its edges, the field will also be given by Eq. (5.5). This is evident because from very close points the surface looks like a very large plane surface with uniform surface charge distribution (of density equal to the local surface charge density).

Example 5.4—Electric field between two parallel charged plates. Now consider two parallel flat plates charged with equal surface charge densities of opposite sign (Fig. 5.4b). If we have in mind the result of the preceding example, superposition yields immediately that between the plates

$$E = \frac{\sigma}{\epsilon_0} \quad (\text{V/m}), \quad (5.6)$$

(Electric field strength between two parallel plates with surface charges σ and $-\sigma$)

and that outside the plates there is no field ($E = 0$). This formula may also seem unimportant for practical cases because it relates to two parallel *infinite* planes. However, this is a good approximation if the plates are of finite size but close to each other with respect to their size.

We will use Eq. (5.6) for the analysis of the parallel-plate capacitor, an important element in electrical engineering.

There are many more electrostatic systems where the magnitude of the electric field strength vector can be obtained by Gauss' law. We will consider several further important practical examples in the next chapters, when we include materials other than air (vacuum) in the analysis.

Questions and problems: Q5.12, P5.5 to P5.20

5.5 Proof of Gauss' Law

Recall that the electric field strength vector \mathbf{E} of any distribution of charge is obtained as a vector sum of individual vectors \mathbf{E} resulting from all point charges of which the charge distribution is composed. Therefore Gauss' law is proven for all cases if we can prove that Eq. (5.4) is valid for a single point charge.

Consider a point charge Q and let us determine the flux of vector \mathbf{E} through a surface element $\mathbf{S} = d\mathbf{S} \mathbf{n}$ (Fig. 5.5). Let us denote this flux by $d\Psi_E$. It is equal to

$$d\Psi_E = \frac{Q}{4\pi\epsilon_0 r^2} dS \cos \alpha = \frac{Q}{4\pi\epsilon_0} \frac{dS_n}{r^2}, \quad = \frac{Q}{4\pi\epsilon_0} \frac{dS_n}{r^2} \quad (5.7)$$

where dS_n is the projection of the flat surface element dS on the plane normal to r .

The projection dS_n can be considered as the base of a cone with the apex at the charge. Let us cut this cone with another plane normal to r , for example at a distance r_1 from the charge, with a base of area dS_1 (Fig. 5.5). From geometry we know that

$$\frac{dS_1}{r_1^2} = \frac{dS_n}{r^2}. \quad (5.8)$$

Note that r_1 is arbitrary. From Eq. (5.7) we conclude that the *flux through any cross-section of the cone is the same*.

Let us now enclose the charge Q in Fig. 5.5 by an arbitrary closed surface S , indicated in the figure. We can divide this surface into elemental surfaces by a very

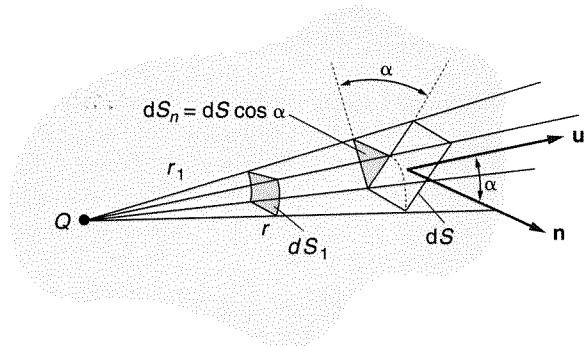


Figure 5.5 A point charge Q and a surface element dS a distance r from it

large number of cones with the common apex at the charge. To calculate the flux through any of these surfaces, we can take *any* cross-section of the cone. Therefore the flux through S is *exactly the same as that through the surface of any sphere centered at the charge*.

The flux through a sphere S of radius r centered at the charge is easy to find. Noting that the angle between the vector \mathbf{E} and the vector surface element $d\mathbf{S}$ of the sphere is zero and that vector \mathbf{E} has the same intensity at all points on S , we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = E \oint_S dS = E4\pi r^2 = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{Q}{\epsilon_0}. \quad (5.9)$$

As explained, because superposition applies, this completes the proof of Gauss' law. Note that the right-hand side of Eq. (5.9) is zero if S does not enclose Q . Therefore the right-hand side in Gauss' law, Eq. (5.4), will be zero if the surface S encloses no charge.

Questions and problems: Q5.13 to Q5.15

5.6 Chapter Summary

1. Gauss' law in Eq. (5.4) is a direct consequence of Coulomb's law, i.e., of the mathematical form of vector \mathbf{E} of a point charge resulting from it. It is, therefore, a fundamental law of electrostatics.
2. In this chapter, Gauss' law has been derived for a system of charges in a vacuum. We know that the elemental charges inside matter, which are the actual sources of the electrostatic field, *are* situated in a vacuum. Therefore Gauss' law, possibly modified, should be applicable to all electrostatic fields, not only in a vacuum.
3. Gauss' law has two important types of applications. One type is proofs of certain general properties of the electrostatic field. The other is the evaluation of the intensity of vector \mathbf{E} of highly symmetrical charge distributions, where we know by symmetry the direction of \mathbf{E} . In such cases Gauss' law, although being a single scalar equation of one ~~scalar~~ **VECTOR** unknown, is sufficient to determine the magnitude of vector \mathbf{E} .

QUESTIONS

- Q5.1.** Prove that in a uniform electric field the flux of the electric field strength vector through any closed surface is zero.
- Q5.2.** Can the closed surface in Gauss' law be infinitesimally small in the mathematical sense? Is the answer different for the case of a vacuum and some other material? Explain.

- Q5.3.** Assume we know that the vector \mathbf{E} satisfies Gauss' law in Eq. (5.4), but we do not know the expression for the vector \mathbf{E} of a point charge. Can this expression be *derived* from Gauss' law? Explain.
- Q5.4.** The center of a small spherical body of radius r , uniformly charged over its surface with a charge Q , coincides with the center of one side of a cube of edge length a ($a > 2r$). What is the flux of the electric field strength vector through the cube?
- Q5.5.** A dielectric cube of edge length a is charged by friction uniformly over its surface, with a surface charge density σ . What is the flux of the electric field strength vector through a slightly smaller and slightly larger imaginary cube? Do the answers look logical? Explain.
- Q5.6.** Is it possible to apply Gauss' law to a large surface enclosing a domain with a number of holes? If you think it is possible, explain how it should be done.
- Q5.7.** Inside an imaginary closed surface S the total charge is zero. Does this mean that at all points of S the vector \mathbf{E} is zero? Explain.
- Q5.8.** A spherical rubber balloon is charged by friction uniformly over its surface. How does the electric field inside and outside the balloon change if it is periodically inflated and deflated to change its radius?
- Q5.9.** Assume that the flux of the electric field strength vector through a surface enclosing a point A is the same for any size and shape of the surface. What does this tell us about the charge at A or in its vicinity?
- Q5.10.** The electric field strength is zero at all points of a closed surface S . What is the charge enclosed by S ?
- Q5.11.** An electric dipole (two equal charges of opposite signs) is located at the center of a sphere of radius greater than half the distance between the charges. What is the flux of vector \mathbf{E} through the sphere?
- Q5.12.** Would it be possible to apply Gauss' law for the determination of the electric field for charged planes with nonuniform charge distribution? Explain.
- Q5.13.** What would be the form of Gauss' law if the unit vector normal to a closed surface were adopted to point into the surface, instead of out of the surface?
- Q5.14.** Gauss' law is a consequence of the factor $1/r^2$ in the expression for the electric field strength of a point charge (i.e., in Coulomb's law). At what step in the derivation of Gauss' law is this the condition for Gauss' law to be valid?
- Q5.15.** Try to derive Gauss' law for a hypothetical electric field where the field strength of a point charge is proportional to $1/r^k$, where $k \neq 2$.

PROBLEMS

- P5.1.** The flux of the electric field strength vector through a closed surface is $100 \text{ V} \cdot \text{m}$. How large is the charge inside the surface?
- P5.2.** A point charge $Q = 2 \cdot 10^{-11} \text{ C}$ is located at the center of a cube. Determine the flux of vector \mathbf{E} through one side of the cube using Gauss' law.
- P5.3.** A point charge $Q = -3 \cdot 10^{-12} \text{ C}$ is $d = 5 \text{ cm}$ away from a circular surface S of radius $a = 3 \text{ cm}$ as shown in Fig. P5.3. Determine the flux of vector \mathbf{E} through S .

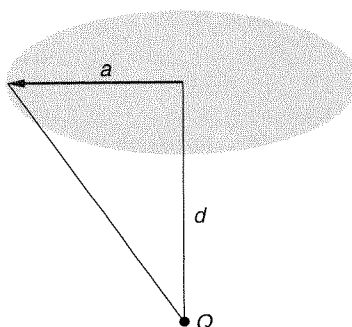


Figure P5.3 A circular surface near a point charge

- P5.4.** Determine the flux of vector \mathbf{E} through a hemispherical surface of radius $a = 5$ cm, if the field is uniform, with $E = 15$ mV/m, and if vector \mathbf{E} makes an angle $\alpha = 30^\circ$ with the hemisphere axis. Use Gauss' law.
- P5.5.** Three parallel thin large charged plates have surface charge densities $-\sigma$, 2σ , and $-\sigma$. Find the electric field everywhere for all combinations of the relative sheet positions and $\sigma = 10^{-6}$ C/m². Do the results depend on the distances between the plates? Determine the equipotential surfaces in all cases, and the potential difference between pairs of plates, if the distance between them is 2 cm.
- P5.6.** A very large flat plate of thickness d is uniformly charged with volume charge density ρ . Find the electric field strength at all points. Determine the potential difference between the two boundary planes, and between the plane of symmetry of the plate and a boundary plane.
- P5.7.** The volume charge density of a thick, very large plate varies as $\rho = \rho_0 x/d$ through the plate, where x is the distance from one of its boundary planes. Find the electric field strength vector everywhere. Plot your result. How large is the potential difference between the two boundary surfaces of the plate? *d IS THE THICKNESS OF THE PLATE*
- P5.8.** Two concentric spherical surfaces, of radii a and $b > a$, are uniformly charged with the same amounts of charge Q , but of opposite signs. Find the electric field strength at all points and present your expressions graphically.
- P5.9.** The spherical surfaces from the previous problem do not have the same charge, but are charged with $Q_{\text{inner}} = 10^{-10}$ C and $Q_{\text{outer}} = -5 \cdot 10^{-11}$ C. The radii of the spheres are $a = 3$ cm and $b = 5$ cm. Find the electric field strength and potential at all points and present your expressions graphically.
- P5.10.** A spherical cloud of radius a has a uniform volume charge of density $\rho = -10^{-5}$ C/m³. Find the electric field strength and potential at all points and present your expressions graphically.
- P5.11.** A spherical cloud shell has a uniform volume charge of density $\rho = 10^{-3}$ C/m³, an inner radius $a = 2$ cm, and an outer radius $b = 4$ cm. Find the electric field strength and potential at all points and present your expressions graphically.
- P5.12.** The volume charge density of a spherical charged cloud is not constant, but varies with the distance from the cloud center as $\rho(r) = \rho_0 r/a$. Determine the electric field strength and potential at all points. Present your results graphically.

- P5.13.** Find the expression for the electric field strength and potential between and outside two long coaxial cylinders of radii a and b ($b > a$), carrying charges Q' and $-Q'$ per unit length. (This structure is known as a coaxial cable, or coaxial line.) Plot your results. Determine the voltage between the two cylinders.
- P5.14.** Repeat problem P5.13 assuming that the two cylinders carry unequal charges per unit length, when these charges are (1) of the same sign, and (2) of opposite signs. Plot your results and compare to problem P5.13.
- P5.15.** A very long cylindrical cloud of radius a has a constant volume charge density ρ . Determine the electric field strength and potential at all points. Present your results graphically. Is it possible in this case to adopt the reference point at infinity? Explain.
- P5.16.** Repeat problem P5.15 assuming that the charge density is not constant, but varies with distance r from the cloud axis as $\rho(r) = \rho_0 r/a$.
- P5.17.** Repeat problem P5.15 assuming that the cloud has a coaxial cavity of radius b ($b < a$) with no charges.
- *P5.18.** Prove that the electric scalar potential cannot have a maximum or a minimum value, except at points occupied by positive and negative charges, respectively.
- *P5.19.** Prove ^{EARNSTHAW'S} Earnshaw's theorem: A stationary system of charges cannot be in a stable equilibrium without external nonelectric forces. (Hint: use the conclusion from problem P5.18.)
- *P5.20.** Prove that the average potential of any sphere S is equal to the potential at its center, if the charge density inside the sphere is zero at every point.