

Suppose that p is a prime number of the form $p = 4k + 3$. The set $I_p = \{ a + bi : p|a \text{ and } p|b \}$ is a maximal ideal of $R = \{ a + bi : a, b \in \mathbb{Z} \}$ (Gaussian integers).

Proof: I_p is an ideal of Gaussian integers because if $a_1 + b_1i$ and $a_2 + b_2i$ are in I_p so is $(a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$ because $(a_1 + a_2)$ and $(b_1 + b_2)$ both would be divisible by p . Also, for any $m + ni : m, n \in \mathbb{Z}$ we have $(m + ni)(a + bi) = (ma - nb) + (an + bm)i$ but since $p|a$ and $p|b$ we have $p|ma - nb$ & $p|an + bm$ therefore I_p is an ideal of Gaussian integers.

To prove that it's maximal, suppose that N is an ideal of R that contains I_p , i.e. $I_p \subset N$. So, there exists $m + ni$ in N that $p \nmid m$ or $p \nmid n$. Without the loss of generality, suppose that $p \nmid n$, i.e. $n \not\equiv 0 \pmod{p}$. We want to show that $p \nmid m^2 + n^2$. Since n is a non-zero element in \mathbb{Z}_p it has a multiplicative inverse that we denote it by n^{-1} , suppose that $p \mid m^2 + n^2$:

$$p \mid m^2 + n^2 \Rightarrow m^2 + n^2 \equiv 0 \pmod{p} \Rightarrow (n^{-1})^2 m^2 + 1 \equiv 0 \pmod{p} \Rightarrow (n^{-1}m)^2 + 1 \equiv 0 \pmod{p}$$

Setting $x = n^{-1}m$ we have $x^2 + 1 \equiv 0 \pmod{p} \Rightarrow x^2 \equiv -1 \pmod{p}$ which has no solutions if p is of the form $p = 4k + 3$. Contradiction! Therefore $p \nmid m^2 + n^2$.

We know that $t = m^2 + n^2 = (m + ni)(m - ni)$, but since $(m + ni) \in N$ and N is an ideal of R , therefore $t \in N$ and $p \nmid t$. According to the Bézout's theorem: $\exists x_0, y_0 \in \mathbb{Z} : px_0 + ty_0 = 1$. But $t \in N$ and N being an ideal implies $ty_0 \in N$, also, $p \in I_p \subset N$ implies $px_0 \in N$, therefore their sum must be in N , that means $1 \in N$. But N being an ideal of R implies that for any $a + bi \in R$, $(a + bi) \cdot 1 \in N$. That proves $N = R$. So I_p is a maximal ideal of R . Therefore the quotient ring R/I_p is a field! (Is it an interesting field in number theory by the way?)