1. System of Particles

Let $Oxyz$ be an inertial frame of reference; $m_1, \dots, m_N$ be a system of particles (point masses) with masses m_i and position vectors $\boldsymbol{r}_i$; and $m = \sum_{i=1}^{N} m_i$ be the total mass of the system.

An external force $\boldsymbol{F}_i$ acts on each particle; additionally, internal forces act within the system: $\boldsymbol{f}_{ij}$ is the force exerted by the i-th particle on the j-th particle ($\boldsymbol{f}_{ij} = -\boldsymbol{f}_{ji}$, $\boldsymbol{f}_{ii} := \boldsymbol{0}$).

The total force acting on the particle m_i is equal to

$$\boldsymbol{G}_i = \boldsymbol{F}_i + \sum_{j=1}^{N} \boldsymbol{f}_{ji}.$$

Position vector of the center of mass:

$$\boldsymbol{r}_S = \frac{1}{m} \sum_{i=1}^{N} m_i \boldsymbol{r}_i.$$

Center of mass motion theorem:

$$m\ddot{\boldsymbol{r}}_S = \boldsymbol{F}, \quad \boldsymbol{F} = \sum_{i=1}^{N} \boldsymbol{F}_i.$$

Here S denotes the center of mass.

The angular momentum of the system relative to point O is defined as the vector $\boldsymbol{K}_O = \sum_{i=1}^{N} m_i (\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i)$.

Angular momentum theorem:

$$\dot{\boldsymbol{K}}_O = \boldsymbol{M}_O,$$

where $\boldsymbol{M}_O = \sum_{i=1}^{N} (\boldsymbol{r}_i \times \boldsymbol{F}_i)$ is the total moment (torque) of external forces about point O.

The kinetic energy of the system is defined as the scalar quantity

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\boldsymbol{r}}_i|^2.$$

Suppose that the forces $\boldsymbol{G}_i$ can be represented as a sum:

$$\boldsymbol{G}_i = \boldsymbol{G}_i^{\perp} + \boldsymbol{Q}_i.$$

Here $\boldsymbol{G}_i^{\perp}$ denote constraint forces for which the relation

$$\sum_{i=1}^{N} (\boldsymbol{G}_i^{\perp}, \dot{\boldsymbol{r}}_i) = 0$$

holds for any motion.

Kinetic energy theorem (work-energy theorem):

$$\dot{T} = \sum_{k=1}^{N} (\dot{\boldsymbol{r}}_k, \boldsymbol{G}_k) = \sum_{i=1}^{N} (\dot{\boldsymbol{r}}_i, \boldsymbol{Q}_i). \tag{1}$$

The sum on the right-hand side of this formula is called the power (rate of work) of the forces $\boldsymbol{Q}_i$.

Rewriting formula (1) in integral form:

$$T\Big|_{t_2} - T\Big|_{t_1} = \int_{t_1}^{t_2} \sum_{i=1}^{N} (\dot{\boldsymbol{r}}_i, \boldsymbol{Q}_i) dt.$$

The integral on the right-hand side of this formula is called the work done by the forces $\boldsymbol{Q}_i$.

If the forces $\boldsymbol{Q}_i$ do not depend explicitly on velocities and time, then the latter integral represents a line integral of a differential form (the "work form"):

$$T\Big|_{t_2} - T\Big|_{t_1} = \int_L \sum_{i=1}^{N} (\boldsymbol{Q}_i, d\boldsymbol{r}_i), \quad \boldsymbol{Q}_i = \boldsymbol{Q}_i(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$$

along the curve

$$L = \{(\boldsymbol{r}_1(t), \ldots, \boldsymbol{r}_N(t)) \in \mathbb{R}^{3N} \mid t \in [t_1, t_2]\}.$$

The curve L may degenerate into a point, or have self-intersections or corners.

The forces $\boldsymbol{Q}_i$ are called conservative (or potential forces) if there exists a function $V = V(t, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ such that

$$\boldsymbol{Q}_i = -\frac{\partial V}{\partial \boldsymbol{r}_i}.$$

The function V is called the potential or potential energy.

The total energy of the system in this case is defined as the function

$$H = T + V.$$

If the forces $\boldsymbol{Q}_i$ are conservative and the potential does not depend explicitly on time, then the total energy is a first integral of the equations of motion:

$$\dot{H} = 0.$$

In the general case, when $\boldsymbol{Q}_i = -\frac{\partial V}{\partial \boldsymbol{r}_i} + \boldsymbol{Q}'_i$, we have

$$\dot{H} = \frac{\partial V}{\partial t} + \sum_{i=1}^{N} (\dot{\boldsymbol{r}}_i, \boldsymbol{Q}'_i).$$

1.1. Theorems of Dynamics in Koenig's Frame. Koenig's frame is defined as a coordinate system $Sx'y'z'$ that translates with its origin located at the center of mass S of the system of particles.

Let $\boldsymbol{\rho}_i$ denote the position vector of the point mass m_i in Koenig's frame, satisfying

$$\sum_{k=1}^{N} m_k \boldsymbol{\rho}_k = 0.$$

The velocity of the point m_i relative to Koenig's frame is $\dot{\boldsymbol{\rho}}_i$.

The angular momentum of the system in Koenig's frame is the vector $\boldsymbol{K}_* = \sum_{i=1}^{N} m_i(\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i)$. The following relation holds:

$$\boldsymbol{K}_O = m(\boldsymbol{r}_S \times \dot{\boldsymbol{r}}_S) + \boldsymbol{K}_*.$$

Angular momentum theorem in Koenig's frame:

$$\dot{\boldsymbol{K}}_* = \boldsymbol{M}_S, \quad \boldsymbol{M}_S = \sum_{i=1}^{N} (\boldsymbol{\rho}_i \times \boldsymbol{F}_i).$$

The kinetic energy in Koenig's frame is defined as

$$T_* = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\boldsymbol{\rho}}_i|^2.$$

Konig's theorem states:

$$T = \frac{m}{2}|\dot{\boldsymbol{r}}_S|^2 + T_*.$$

Kinetic energy theorem in Koenig's frame:

$$\dot{T}_* = \sum_{i=1}^{N}(\dot{\boldsymbol{\rho}}_i, \boldsymbol{G}_i).$$

2. Rigid Body

Suppose that the system of particles $m_1, \ldots, m_N$ forms a rigid body.

A rigid body is defined as a system of particles such that:

(1) the system contains at least three non-collinear points;
(2) the distances between the particles are kept constant by internal forces.

In this case,

$$\sum_{k=1}^{N}(\dot{\boldsymbol{r}}_k, \boldsymbol{G}_k) = \sum_{k=1}^{N}(\dot{\boldsymbol{r}}_k, \boldsymbol{F}_k), \quad \sum_{k=1}^{N}(\dot{\boldsymbol{\rho}}_k, \boldsymbol{G}_k) = \sum_{k=1}^{N}(\dot{\boldsymbol{\rho}}_k, \boldsymbol{F}_k).$$

The angular momentum of a rigid body in Koenig's frame is calculated by the formula

$$\boldsymbol{K}_* = J_S\boldsymbol{\omega},$$

where J_S is the inertia operator (inertia tensor) of the rigid body about the center of mass S, and $\boldsymbol{\omega}$ is the angular velocity of the rigid body relative to the inertial frame.

Hereafter, the time derivative of the angular velocity vector $\dot{\boldsymbol{\omega}}$ is taken relative to the inertial frame; however, it is useful to recall that the derivative of the angular velocity vector with respect to the inertial frame coincides with its relative derivative in the body-fixed frame:

$$\dot{\boldsymbol{\omega}} = \frac{\delta}{\delta t}\boldsymbol{\omega} + (\boldsymbol{\omega} \times \boldsymbol{\omega}) = \frac{\delta}{\delta t}\boldsymbol{\omega}.$$

Thus,

$$\frac{d}{dt}(J_S\boldsymbol{\omega}) = \frac{\delta}{\delta t}(J_S\boldsymbol{\omega}) + (\boldsymbol{\omega} \times J_S\boldsymbol{\omega}) = J_S\frac{\delta}{\delta t}\boldsymbol{\omega} + (\boldsymbol{\omega} \times J_S\boldsymbol{\omega}) = J_S\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times J_S\boldsymbol{\omega}).$$

Angular momentum theorem for a rigid body in Koenig's frame [**1**]:

$$J_S\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times J_S\boldsymbol{\omega}) = \boldsymbol{M}_S. \tag{2}$$

Kinetic energy of a rigid body in Koenig's frame:

$$T_* = \frac{1}{2}(\boldsymbol{\omega}, J_S\boldsymbol{\omega}) = \frac{1}{2}(\boldsymbol{\omega}, \boldsymbol{K}_*). \tag{3}$$

Kinetic energy theorem for a rigid body in Koenig's frame:

$$\dot{T}_* = (\dot{\boldsymbol{\rho}}_A, \boldsymbol{F}) + (\boldsymbol{\omega}, \boldsymbol{M}_A), \quad \boldsymbol{M}_A = \sum_{i=1}^{N}(\boldsymbol{Am_i} \times \boldsymbol{F}_i), \quad \forall A \in \text{rigid body}.$$

If $A = S$, then $\dot{\boldsymbol{\rho}}_A = 0$ and

$$\dot{T}_* = (\boldsymbol{\omega}, \boldsymbol{M}_S).$$

Kinetic energy theorem for a rigid body:

$$\dot{T} = (\dot{\boldsymbol{r}}_A, \boldsymbol{F}) + (\boldsymbol{\omega}, \boldsymbol{M}_A), \quad \forall A \in \text{rigid body}.$$

2.1. Rigid Body with a Fixed Point. Assume that the origin O is a point fixed in the rigid body. Then the angular momentum of the rigid body about point O is given by $\boldsymbol{K}_O = J_O\boldsymbol{\omega}$.

Angular momentum theorem for a rigid body about point O:

$$J_O\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times J_O\boldsymbol{\omega}) = \boldsymbol{M}_O.$$

The kinetic energy of the rigid body is computed as

$$T = \frac{1}{2}(\boldsymbol{\omega}, J_O\boldsymbol{\omega}).$$

Kinetic energy theorem for a rigid body with a fixed point:

$$\dot{T} = (\boldsymbol{\omega}, \boldsymbol{M}_O).$$

2.2. Planar Motion of a Rigid Body.

Definition 1. *A rigid body is said to undergo planar (plane-parallel) motion if there exists a fixed plane* Π *such that the velocities of all points of the rigid body remain parallel to* Π *throughout the motion.*

In this case, the angular velocity of the rigid body is perpendicular to Π:

$$\Pi: \quad \boldsymbol{\omega} = \dot{\varphi}\boldsymbol{n}, \quad |\boldsymbol{n}| = 1, \quad \boldsymbol{n} \perp \Pi.$$

Here φ is the angle of rotation of the rigid body.

Taking the dot product of formula (2) with $\boldsymbol{n}$ yields

$$I_{\mathscr{S}}\ddot{\varphi} = (\boldsymbol{M}_S, \boldsymbol{n}),$$

where $I_{\mathscr{S}}$ is the moment of inertia of the rigid body about the axis passing through the center of mass perpendicular to the plane Π.

Formula (3) takes the form

$$T_* = \frac{1}{2}I_{\mathscr{S}}\dot{\varphi}^2.$$

3. Less Common Formulas

Relation between moments about different points:

$$\boldsymbol{M}_B = \sum_{i=1}^{N}(\boldsymbol{Bm_i} \times \boldsymbol{F}_i), \quad \boldsymbol{M}_A = \sum_{i=1}^{N}(\boldsymbol{Am_i} \times \boldsymbol{F}_i)$$

$$\Longrightarrow \boldsymbol{M}_A = (\boldsymbol{AB} \times \boldsymbol{F}) + \boldsymbol{M}_B, \quad \boldsymbol{F} = \sum_{i=1}^{N}\boldsymbol{F}_i.$$

Let A be an arbitrarily moving point; denote

$$\boldsymbol{K}_A = \sum_{i=1}^{N} m_i(\boldsymbol{Am}_i \times \boldsymbol{v}_i), \quad \boldsymbol{v}_i = \dot{\boldsymbol{r}}_i.$$

The following relations hold: $\boldsymbol{K}_O = m(\boldsymbol{OA} \times \boldsymbol{v}_S) + \boldsymbol{K}_A, \quad \boldsymbol{K}_S = \boldsymbol{K}_*$, and

$$\dot{\boldsymbol{K}}_A + m(\boldsymbol{v}_A \times \boldsymbol{v}_S) = \boldsymbol{M}_A. \tag{4}$$

If the system of particles is a rigid body and point A is attached to this rigid body, then

$$\boldsymbol{K}_A = J_A\boldsymbol{\omega} + m(\boldsymbol{AS} \times \boldsymbol{v}_A),$$

and formula (4) takes the form [**1**]:

$$J_A\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times J_A\boldsymbol{\omega}) + m(\boldsymbol{AS} \times \boldsymbol{a}_A) = \boldsymbol{M}_A.$$

4. Inertia Operator

DEFINITION 2. *The inertia operator (inertia tensor) at point O is defined by*

$$J_O\boldsymbol{x} = \sum_{k=1}^{N} m_k(\boldsymbol{r}_k \times (\boldsymbol{x} \times \boldsymbol{r}_k)); \quad J_O : \mathbb{R}^3 \to \mathbb{R}^3.$$

Let S denote the center of mass of the rigid body.

THEOREM 1. *The inertia operator is symmetric and positive definite.*

The theorem follows from the formula

$$(\boldsymbol{y}, J_O\boldsymbol{x}) = \sum_{k=1}^{N} m_k(\boldsymbol{x} \times \boldsymbol{r}_k, \boldsymbol{y} \times \boldsymbol{r}_k), \tag{5}$$

which, in turn, is obtained by cyclic permutation in the scalar triple product.

In proving the positive definiteness of the inertia operator, essential use is made of the fact that, by definition of a rigid body, not all points $m_1, \dots, m_N$ lie on a single line.

Inertia ellipsoid: The ellipsoid associated with the rigid body given by

$$\{\boldsymbol{x} \in \mathbb{R}^3 \mid (\boldsymbol{x}, J_O\boldsymbol{x}) = b^2 \neq 0\}.$$

DEFINITION 3. *Let the axis $\mathscr{L}$ pass through point O parallel to a unit vector $\boldsymbol{e}$. The moment of inertia of the rigid body about the axis $\mathscr{L}$ is the scalar $I_{\mathscr{L}} = (\boldsymbol{e}, J_O\boldsymbol{e})$.*

It follows from formula (5) that $I_{\mathscr{L}} = \sum_{k=1}^{N} m_k d_k^2$, where d_k is the distance from m_k to the line $\mathscr{L}$.

DEFINITION 4. *Let the axis $\mathscr{L}$ pass through point O parallel to a unit vector $\boldsymbol{e}$. The axis $\mathscr{L}$ is called a principal axis of the operator J_O if the vector $\boldsymbol{e}$ is an eigenvector of J_O.*

It is easy to see that the corresponding eigenvalue is $I_{\mathscr{L}}$, i.e., $J_O\boldsymbol{e} = I_{\mathscr{L}}\boldsymbol{e}$.

DEFINITION 5. *Let the axis $\mathscr{L}$ pass through point O parallel to a unit vector $\boldsymbol{e}$. The axis $\mathscr{L}$ is called a principal central axis of the inertia operator J_O if it is a principal axis of J_O and contains the center of mass of the rigid body ($S \in \mathscr{L}$).*

THEOREM 2 (Huygens–Steiner Theorem). *The following formula holds:*

$$J_O\boldsymbol{x} = m(\boldsymbol{OS} \times (\boldsymbol{x} \times \boldsymbol{OS})) + J_S\boldsymbol{x}. \tag{6}$$

PROOF. Let $\boldsymbol{\rho}_k$ denote the position vector of the point mass m_k originating from point S:

$$\sum_{k=1}^{N} m_k\boldsymbol{\rho}_k = 0,$$

and let $\boldsymbol{r}_k$ be the position vector of m_k originating from point O. Now, formula (6) is obtained by substituting the expression $\boldsymbol{r}_k = \boldsymbol{OS} + \boldsymbol{\rho}_k$ into the definition of J_O. □

We note two corollaries of the Huygens–Steiner theorem.

Corollary 1. *Let the axis $\mathscr{L}$ pass through point O parallel to a unit unit vector $\boldsymbol{e}$ and be a principal central axis of the operator J_O. Then it is a principal central axis of the operator J_A for any point $A \in \mathscr{L}$.*

Corollary 2. *Let the axis $\mathscr{L}$ pass through point O parallel to a unit vector $\boldsymbol{e}$, and let the axis $\mathscr{N}$ pass through point S parallel to the unit vector $\boldsymbol{e}$. Suppose the distance between the axes is d. Then the following formula holds:*

$$I_{\mathscr{L}} = I_{\mathscr{N}} + md^2.$$

Theorem 3. *Suppose that the points $m_1, \ldots, m_N$ are arranged symmetrically with respect to a plane Π. Let $A \in \Pi$ be an arbitrary point belonging to this plane, and let $\boldsymbol{n}$ be the unit normal to the plane.*

Then the axis $\mathscr{A}$ passing through point A along the vector $\boldsymbol{n}$ is a principal axis of inertia of the operator J_A, i.e., $J_A\boldsymbol{n} = I_{\mathscr{A}}\boldsymbol{n}$.

Proof. Indeed, let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection operator with respect to the plane Π.

Notice that for any two vectors $\boldsymbol{x}, \boldsymbol{y}$, the identity $P(\boldsymbol{x} \times \boldsymbol{y}) = -(P\boldsymbol{x} \times P\boldsymbol{y})$ holds. Consequently,

$$PJ_A\boldsymbol{x} = \sum_{k=1}^{N} m_k(P\boldsymbol{r}_k \times (P\boldsymbol{x} \times P\boldsymbol{r}_k)).$$

In this formula, the position vectors $\boldsymbol{r}_j$ originate from point A. Due to the symmetric arrangement of the masses,

$$\sum_{k=1}^{N} m_k(P\boldsymbol{r}_k \times (P\boldsymbol{x} \times P\boldsymbol{r}_k)) = \sum_{k=1}^{N} m_k(\boldsymbol{r}_k \times (P\boldsymbol{x} \times \boldsymbol{r}_k)) = J_A P\boldsymbol{x}.$$

Thus, $PJ_A = J_A P$. Since $P\boldsymbol{n} = -\boldsymbol{n}$ and the eigenspace corresponding to the eigenvalue -1 is one-dimensional, we obtain

$$PJ_A\boldsymbol{n} = J_A P\boldsymbol{n} = -J_A\boldsymbol{n} \Longrightarrow J_A\boldsymbol{n} = \lambda\boldsymbol{n}.$$

□

References

[1] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart, 1977.