Note:

I was prompted to write this when I noticed in class that the General Constitutive Law for Newtonian fluids is commonly written as

$$\tau = \mu\left[\vec{\nabla}\vec{u} + \left(\vec{\nabla}\vec{u}\right)^T\right].$$

The first thing I thought when I saw that was "you can't just take the gradient of a vector!". I decided to write it out in a more rigorous way (in Cartesian Coordinates). I would appreciate any feedback on any errors that may be here.

The General Constitutive Law:

Define μ to be the coefficient of viscosity for the fluid. This coefficient μ can be a function of pressure and temperature, $\mu(p,T)$.

We suppose that $\vec{u} : \boldsymbol{R}^3 \times (\boldsymbol{R}^+ \cup \{0\}) \to \boldsymbol{R}^3$ is the velocity of the flow defined at every point $\vec{x} = (x,y,z)$ in the 3D domain D (or $\boldsymbol{R}^3$) and every point $t \in (\boldsymbol{R}^+ \cup \{0\})$. There exist functions u_x, u_y, and u_z such that this velocity function $\vec{u}$ has the form

$$\vec{u} = (u_x(x,y,z,t), u_y(x,y,z,t), u_z(x,y,z,t))$$

for each $(x,y,z,t) \in (\boldsymbol{R}^3 \times (\boldsymbol{R}^+ \cup \{0\}))$

Then, we define J to signify the Jacobian matrix, we have

$$\begin{aligned}\vec{\nabla}\vec{u} &= J(\vec{u}) \\ &= \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}\end{aligned}$$

for each $(x,y,z) \in D$.

For common fluids, the General Constitutive Law says that the stress tensor τ is given by

$$
\begin{aligned}
\tau &= \mu\left[\vec{\nabla}\vec{u} + \left(\vec{\nabla}\vec{u}\right)^T\right] \\
&= \mu\left[J(\vec{u}) + (J(\vec{u}))^T\right] \\
&= \mu\left[\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}^T\right] \\
&= \mu\left[\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} + \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix}\right] \\
&= \mu\begin{pmatrix} 2\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & 2\frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} & \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} & 2\frac{\partial u_z}{\partial z} \end{pmatrix}
\end{aligned}
$$

The general constitutive law can not be derived or proven from any mathematical theories. It is simply an experimentally observed fact of life for Newtonian fluids.