observer (z, Z) but no distinguished representative in $[\lambda]$; (b) given $\lambda \in [\lambda]$ but no distinguished Z; (c) given both. Answers: (a) The spatial direction (z, Z) measures for $[\lambda]$; often this is the most important measurement. (b) Only the energy-momentum λ_*. (c) Energy, 3-momentum, frequency, wavelength, color, spatial direction.

5.2 Light signals

Define a *light signal* as an equivalence class (Section 5.0.2) of freely falling photons; the motivations are indicated in Section 5.1.1 and Exercise 5.1.7b.

EXAMPLE 5.2.1. Let $(\mathbb{R}^4, \boldsymbol{g})$ be Minkowski space. Each inextendible light signal has exactly one representative $\lambda: \mathbb{R} \to \mathbb{R}^4$ of the form $\lambda u = (v, 0) + u(w, 1)$ where $v, w \in \mathbb{R}^3$ and $w \cdot w = 1$ ($\mathbb{R}^3$ inner product). Conversely, $\forall$ pair $(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $w \cdot w = 1$ there is a unique light signal, say $[\lambda]_{(v,w)}$. For example one here has a new equivalence relation among the light signals themselves: $[\lambda]_{(v,w)} \sim [\lambda]_{(v',w')}$ iff $w = w'$ and $(v - v') \cdot w = 0$.

Two light signals are said to meet at infinity iff they are equivalent in this sense. This concept can be extended to certain kinds of nonflat spacetimes and is then very useful (Hawking–Ellis [1]).

EXAMPLE 5.2.2. Let $(M, \boldsymbol{g})$ be Einstein–de Sitter spacetime (Section 1.4). Thus $M = \mathbb{R}^3 \times (0, \infty)$, $\boldsymbol{g} = (u^4)^{4/3} \sum_{\mu=1}^{3} \boldsymbol{du}^\mu \otimes \boldsymbol{du}^\mu - \boldsymbol{du}^4 \otimes \boldsymbol{du}^4$. There exist light signals lying entirely within the $(0, 0, u^3, u^4)$ plane (Exercise 5.0.8). As inspection of $\boldsymbol{g}$ suggests, analyzing one of these suffices to give complete information on all freely falling photons in $(M, \boldsymbol{g})$ (Exercise 5.2.5 following).

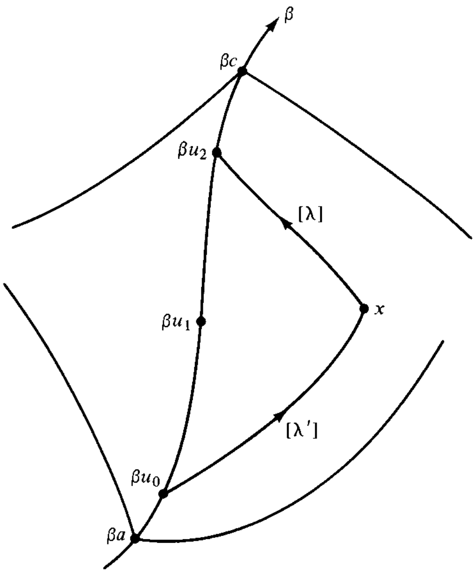
Define $\lambda: (0, \infty) \to M$ by $\lambda u = (0, 0, 3u^{1/5}, u^{3/5})$. We claim λ is an inextendible, freely falling photon.

PROOF. λ is C^∞ and future pointing. Moreover, $\boldsymbol{g}(\lambda_*, \lambda_*) = (u^4 \circ \lambda)^{4/3} \times [\boldsymbol{du}^3(\lambda_*)]^2 - [\boldsymbol{du}^4(\lambda_*)]^2$. Now $u^4\lambda u = u^{3/5}$, $[\boldsymbol{du}^3(\lambda_*)](u) = (3/5)u^{-4/5}$ and $[\boldsymbol{du}^4(\lambda_*)](u) = (3/5)u^{-2/5}$. Thus $\boldsymbol{g}(\lambda_*, \lambda_*) = 0$; thus λ is a photon. To see that λ is geodesic, note that ∂_3 is a Killing vector field (Section 3.6.3) and $\boldsymbol{g}(\lambda_*, \partial_3) = (u^4 \circ \lambda)^{4/3}[\boldsymbol{du}^3(\lambda_*)] = 3/5$, a constant. By Exercise 5.0.9, this suffices. Finally, λ is inextendible since the scalar curvature obeys $S\lambda u \to \infty$ as $u \to 0$. □

We define λ as *the standard photon* on $(M, \boldsymbol{g})$, its equivalence class (Section 5.0.2) as *the standard light signal.*

Let $\beta: \mathscr{E} \to M$ be an observer on a spacetime M. From which points can β receive light signals and to which ones can he send them? For Minkowski space the situation is simple (Exercise 5.1.5); for Einstein–de Sitter spacetime it is a little more complicated (Exercise 5.1.6); in a black hole situation there are points that can receive light signals from "outside" observers, but cannot send any (Section 7.5); and so on. Locally, the situation remains simple, as the following proposition shows.

Proposition 5.2.3. *Suppose $u_1 \in \mathscr{E}$ is given. There exists an open interval $\mathscr{F} \subset \mathscr{E}$ containing u_1 and an open neighborhood $\mathscr{W}$ of βu_1 such that, $\forall x \in \mathscr{W} - \beta\mathscr{F}$ the following holds. There exist $u_0, u_2 \in \mathscr{F}$, a light signal $[\lambda]$ from x to βu_2 and a light signal $[\lambda']$ from βu_0 to x; u_0, u_2, $[\lambda]$ and $[\lambda']$ are unique.*



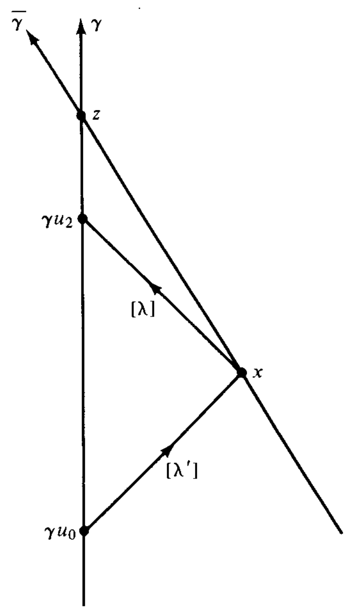
PROOF. Let $\mathscr{U}$ be a simply convex open neighborhood of βu_1, $[a, c]$ be an interval in $\mathscr{E}$ containing u_1 as an interior point such that $\beta[a, c] \subset \mathscr{U}$.

Regard $(\mathscr{U}, g)$ as a spacetime as in Sections 5.0.5 and 5.0.6, and adopt the notation of those subsections.

Let $\mathscr{W} = (I_{\beta c}{}^-) \cap (I_{\beta a}{}^+)$. Thus $\mathscr{W}$ is an open neighborhood of $\beta(a, c)$, in particular of βu_1 (Exercise 5.0.10). Define $\mathscr{F} = (a, c)$. We claim $\mathscr{W}$ and $\mathscr{F}$ have the required property.

In fact, suppose $x \in \mathscr{W} - \beta\mathscr{F}$. Then $\beta c \in I_x{}^+$ since $x \in I_{\beta c}{}^-$. Similarly $\beta a \in I_x{}^-$, in particular $\beta a \notin \text{closure } I_x{}^+$. Since β is continuous, the lemma in Section 5.0.6 shows there exists a unique $u_2 \in (a, c)$ such that the geodesic λ: $[0, 1] \to \mathscr{U}$ from x to βu_2 is future pointing lightlike. Thus u_2, $[\lambda]$ exist and are unique. The dual argument proves u_0, $[\lambda']$ exist and are unique. □

EXAMPLE 5.2.4. RADAR. Let (M, g) be two-dimensional Minkowski space (Section 0.2), γ: $\mathbb{R} \to M$ be the observer defined by $\gamma u = (0, u)$. For l, $\theta \in (0, \infty)$, $\bar{\gamma}$: $\mathbb{R} \to M$, defined by $\bar{\gamma} u = (l - u \sinh \theta, u \cosh \theta)$, is another observer. γ can use radar to observe $\bar{\gamma}$ as follows. At proper time u_0, γ emits a light signal $[\lambda']$ which travels to $x \in \bar{\gamma}\mathbb{R}$, is "reflected" there, and returns to γ at proper time u_2 as a light signal $[\lambda]$. Working algebraically, the reader can check that if $[\lambda']$ is emitted prior to the collision event z, $u_2 = u_0 e^{-2\theta} + 2le^{-\theta}\cosh\theta$. Thus if γ knows *a priori* that $\bar{\gamma}$ is freely falling, two radar measurements (u_0, u_2), (u_0', u_2') suffice to fix l and θ and thus to determine the world line of $\bar{\gamma}$.



In more general cases γ may need a whole function $u_2(u_0)$ and must also observe directions (Section 5.1.3). Since meter sticks do not really make sense in general relativity and are not used in actual astronomical measurements, one regards a radar set as the basic distance measuring device (cf. Exercise 2.3.12d and the comment below it).

Exercise 5.2.5

Let (M, g) be Einstein–de Sitter spacetime; thus $M = \mathbb{R}^3 \times (0, \infty)$. Show: (a) If $\bar{\psi}: \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry, $\psi: M \to M$, defined by $\psi(v, t) = (\bar{\psi}v, t)$, is an isometry. (b) If $\gamma: \mathscr{E} \to M$ is a geodesic there is an isometry $\psi: M \to M$ such that $\gamma \circ \psi$ lies in the plane $u^1 = 0 = u^2$. (c) If γ in (b) is an inextendible, freely falling photon there exists a ψ such that $[\gamma \circ \psi] = [\lambda]$, λ as in Example 5.2.2. (Hint: See Section 6.0.5 if you get stuck.)

Exercise 5.2.6 Orthogonality and radar

Let $\gamma: \mathscr{E} \to M$ be an observer, $u_1 \in \mathscr{E}$, $\mathscr{W}$ be a neighborhood of γu_1 with the properties in Proposition 5.2.3, $\alpha: [0, a) \to \mathscr{W}$ be a curve which intersects γ's world line at the parameter value 0 with $\alpha 0 = \gamma u_1$. Define a function $f: (0, a) \to \mathbb{R}$ by $f = u_1 - \frac{1}{2}(u_2 + u_0)$, where u_2 and u_0 are as in Proposition 5.2.3 with $x = \alpha s$

$\forall s \in (0, a)$. (a) Suppose M is Minkowski spacetime and γ, α are geodesics. Show $f = 0$ iff γ and α are orthogonal. (b) Dropping the geodesic restriction on both show that $\lim_{s \to 0} f(s) = 0$ and show that $\lim_{s \to 0} f'(s) = 0$ iff $g(\gamma_* 1, \alpha_* 0) = 0$. (c) Show that (b) remains valid when M is an arbitrary spacetime.

Exercise 5.2.7

(a) Given $x \in M$, show there exists an open neighborhood $\mathscr{U}$ of x with the following property. $\forall z \in \mathscr{U}$, there exists $y \in \mathscr{U}$ such that there exists a lightlike geodesic from x to y and there exists a lightlike geodesic from y to z. (b) Suppose $N \subset M$ is an open proper submanifold. Show there exists a point x on the boundary of N. (c) By combining (a) and (b), prove Proposition 1.3.2. (d) Use Proposition 1.3.2, Example 5.2.2, and Exercise 5.2.5 to show Einstein–de Sitter spacetime is maximal.

5.3 Synchronizable reference frames

In Section 2.3 we defined synchronizable reference frames without making explicit the empirical significance of synchronizability. Now that we have light signals at our disposal, we can fill in this gap. Throughout the section Z is a reference frame on spacetime M and ζ is the physically equivalent 1-form.

Recall the following results from Section 2.3: Z is locally proper time synchronizable iff Z is geodesic and irrotational, proper time synchronizable iff there is a function $t: M \to \mathbb{R}$ such that $\zeta = -dt$. Moreover, suppose Z is proper time synchronizable and t is as above. (a) Each level surface of t is a 3-manifold everywhere orthogonal to Z. (b) The function t is unique up to a single additive constant. (c) Let $\gamma: \mathscr{E} \to M$ be an inextendible integral curve of Z; thus γ is an observer in Z. Then there is a constant $a_\gamma \in \mathbb{R}$ such that $t\gamma u = u - a_\gamma$—that is, $t \circ \gamma$ agrees with proper time u up to a constant. Hence the term proper time function for t in Section 2.3.

If Z is proper time synchronizable, a set $\{\gamma\}$ of observers in Z can empirically synchronize their clocks, by radar, as follows. Suppose first that by sheer luck the observers already have $a_\gamma = 0 \, \forall \gamma$ and for one t. Consider any two sufficiently nearby observers $\gamma, \gamma' \in \{\gamma\}$. Here "sufficiently nearby" means Proposition 5.2.3 is applicable. It also means so nearby that the approximation $u_1' = \frac{1}{2}(u_0 + u_2)$ below holds to within other empirical inaccuracies.

Since we are discussing actual measurements the standard mathematics vs. physics ambiguity (Section 2.1.2) on infinitesimals here comes into play and Einstein's comment applies. To make the discussion mathematically rigorous, we would need to assume an infinite number of observers present. Rather than adopt this wildly unrealistic approach we couch the discussion in physicists' language.

Suppose γ and γ' communicate by radar. Say γ emits his light signal at his proper time $u_0 = t\gamma u_0$, the signal strikes γ' at his proper time $u_1' = t\gamma' u_1'$ and returns to γ at $u_2 = t\gamma u_2$ (cf. the figure in Proposition 5.2.3). By the orthogonality property (a) above and Exercise 5.2.6, u_1' is the average:

$u_1' = \frac{1}{2}(u_0 + u_2)$ within the limit of empirical accuracy. By subsequent communication γ and γ' learn that this *consistency condition* held. Now imagine each observer in $\boldsymbol{Z}$ continually performing measurements of this kind with each nearby neighbor. By noting that the consistency condition always holds in this large set of interlocking measurements they can conclude that orthogonal surfaces t = constant must exist and that t is indeed a proper time function.

In the more general case that $a_{\gamma'} \neq 0$ for some of the observers in $\boldsymbol{Z}$, all that is required is that one observer, say γ, take the lead. γ autocratically decides that $t\gamma u = u$. He transmits his fiat to his neighbors—for example, by demanding $t\gamma' u_1'$ above be given by $t\gamma' u_1' = \frac{1}{2}(u_0 + u_2)$. They in turn inform their neighbors, and so on. This determines a function t on (part of) M. Somewhat angry at first, the other observers find that at worst they need only change the origin of their own proper time to get agreement with t and that now all further radar measurements as above give consistency.

Exercise 5.3.1

Generalize the above discussion to the case that $\boldsymbol{Z}$ is synchronizable but not proper time synchronizable by assuming one autocrat γ and other observers who regard the consistency condition, $t_1 = \frac{1}{2}(t_0 + t_2)$ $\forall$ nearby radar measurement, as more important than insisting on their own proper time.

In practice, γ may have some justification—for example, as the "observer" in the center of a star or as an "observer at infinity." Note that unless the chronological future $\{z \in M \mid x \ll z$ for some x on γ's world line$\}$ is all of M, t is not radar determined on all of M by signals that travel no faster than light.

Exercise 5.3.2

Take a rotating reference frame on Minkowski spacetime and show explicitly how the radar measurements can lead to an inconsistency with the assumption that a time function exists (cf. Exercise 2.3.15).

5.4 Frequency ratio

Suppose we observe a photon from a star. By measuring its energy we can assign it a frequency (Section 5.1). Suppose we see first one photon, then another, then another, and so on. By counting how many photons arrive during one second of our own proper time we get a conceptionally different "frequency": number per unit proper second. We now show these two concepts are consistent. In practice, both kinds of measurement are key tools in analyzing, roughly speaking, "how fast the star is moving" or "how much gravity is between us and it."

Formally, the situation of interest is the following. We have a rectangle $\mathscr{D} = [a, b] \times [-\varepsilon, \varepsilon]$ and a map $\sigma: \mathscr{D} \to M$ as in Section 5.0.3. $\forall v \in [-\varepsilon, \varepsilon]$ the curve $\sigma_v: [a, b] \to M$ is a freely falling photon ("from the star to us").