duced in Section 0.0.8, Q is a *geodesic reference frame* iff in addition $D_Q Q = 0$. Let $\mathbf{Q}$ be a reference frame in the rest of this section.

The integral curves of Q are called *observers in* Q. All observers in a geodesic reference frame are freely falling. Let ω be the 1-form physically equivalent to Q. If $\gamma: \mathscr{E} \to M$ is an observer in Q, then $du = -\gamma^*\omega$ since $(\gamma^*\omega)(d/du) = \omega(\gamma_*) = g(Q, \gamma_*) = g(\gamma_*, \gamma_*) = -1$. Thus geometric properties of ω can be related to the proper times of the observers in Q, as in the following discussion.

Q is called: *locally synchronizable* iff $\omega \wedge d\omega = 0$, *locally proper time synchronizable* iff $d\omega = 0$, *synchronizable* iff there are C^∞ functions h and t on M such that $h > 0$ and $\omega = -hdt$, and *proper time synchronizable* iff $\omega = -dt$. Since $-hdt \wedge d(-hdt) = 0$, a synchronizable reference frame is locally synchronizable; similarly $\omega = -dt \Rightarrow d\omega = 0$, so that a proper time synchronizable reference frame is locally proper time synchronizable. Conversely, if Q is locally synchronizable (respectively, locally proper time synchronizable) then the restriction of Q to every sufficiently small open set is a synchronizable (respectively, a proper time synchronizable) reference frame (Exercise 2.3.11). If Q is synchronizable or proper time synchronizable, respectively, any function t as above is called a *time function* or a *proper time function for* Q, respectively. If a time function exists, it is not unique. If a proper time function exists, it obeys $du = \gamma^*dt$ $\forall$ observer γ in Q.

If Q is synchronizable, then all the level hypersurfaces of the time function t are orthogonal to Q, and hence also orthogonal to all the observers in Q. Conversely, if Q is an arbitrary reference frame on M and t is a function on M such that (1) dt is nowhere zero and (2) the level hypersurfaces of t are everywhere orthogonal to Q, then Q is synchronizable and $\pm t$ is a time function for Q.

From a mathematical point of view, the terminology used in the above definitions is easily understood: If Q is proper time synchronizable, let $\mathscr{N}_a$ be the level hypersurface of a proper time function t for Q defined by $t = a$. For simplicity, let us consider only the case where each observer in Q meets each $\mathscr{N}_a$ exactly once. Then every observer $\gamma: \mathscr{E} \to M$ in Q can adjust his "atomic clock" $\mathscr{E}$ so that his proper time is 0 when his world line intersects $\mathscr{N}_0$. Since $du = \gamma^*dt$, it follows that when the proper time of each observer in Q is a, his world line intersects $\mathscr{N}_a$. In this way, a proper time function for a proper time synchronizable Q achieves a uniform synchronization among all observers in Q. If Q is only synchronizable but not proper time synchronizable and t is a time function for Q, then $du = (h \circ \gamma)\gamma^*dt$ for each observer γ in Q. Since $h \circ \gamma$ is not identically equal to 1, $t \circ \gamma$ no longer equals u up to an additive constant, but is nevertheless explicitly expressible in terms of u by $t \circ \gamma = \int (du/h\gamma u)$. Thus if each observer agrees to use a modified time, cooperation among observers becomes comparatively convenient.

From a physical point of view, the reason for the above terminology is somewhat more profound. Using photons, which will be systematically discussed in Chapter 5, we shall show in Section 5.3 how the observers in a