Conservation of Energy

$$T + U = constant$$

$$\frac{m(\dot{x}^2 + \dot{y}^2)}{2} - \frac{GMm}{r} = constant$$

$$x = r\,cos(\theta), y = r\sin(\theta)$$

$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta} \text{ (1)}$$

$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta} \text{ (2)}$$

$$m\frac{\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right)^2 + \left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right)^2}{2} - \frac{GMm}{r} = 0$$

$$\tfrac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2\right) - \tfrac{GMm}{r} = constant \text{ (1)}$$

Conservation of Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p} = constant$$

$$r = \hat{\imath}\, r\cos(\theta) + \hat{\jmath}\, r\sin(\theta) + \hat{k}\, 0$$

Using (1) and (2)

$$p = \hat{\imath}\, m\dot{x} + \hat{\jmath} m\dot{y} = \hat{\imath}\, m\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right) + \hat{\jmath} m\left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right)$$

$$\vec{L} = \vec{r} \times \vec{p} = \ \widehat{k}\,[r\cos(\theta)\ \ m\left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right) - \ r\sin(\theta)\ \ m\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right)]$$

Simplifying yields

$$\vec{L} = mr^2\dot{\theta} = constant$$

$$\dot{\theta} = \frac{L}{mr^2}$$

Substituting this in (1)

$$\frac{1}{2}\left(m\dot{r}^2 + mr^2\frac{L^2}{m^2r^4}\right) - \frac{GMm}{r} = constant$$

$$\tfrac{1}{2}\left(m\dot{r}^2 + \tfrac{L^2}{mr^2}\right) - \tfrac{GMm}{r} = constant \text{ (3)}$$

Substitute $r = \frac{1}{u(\theta)}$

$$\dot{r} = \frac{d}{dt}\left[u^{-1}\big(\theta(t)\big)\right] = -u\big(\theta(t)\big)^{-2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

Using this in (3) and differentiating with respect to time

$$\frac{d}{dt}\left[\frac{1}{2}\left(mu^{-4}\frac{du}{d\theta}^{2}\frac{d\theta}{dt}^{2} + \frac{L^2}{m}u^2\right) - GMmu\right] = constant$$

$$\frac{1}{2}\left(m(-4)u^{-5}\frac{du}{d\theta}\frac{d\theta}{dt}\frac{du}{d\theta}^{2}\frac{d\theta}{dt}^{2} + mu^{-4}(2)\frac{du}{d\theta}\frac{d\theta}{dt}\frac{d\theta}{dt}^{2} + mu^{-4}\frac{du}{d\theta}^{2}(2)\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{L^2}{m}(2)u\frac{du}{d\theta}\frac{d\theta}{dt}\right) - GMm\frac{du}{d\theta}\frac{d\theta}{dt} = 0$$

Simplifying and multiplying by u^5

$$-2\,m\frac{du}{d\theta}^{3}\frac{d\theta}{dt}^{3} + mu\frac{du}{d\theta}\frac{d\theta}{dt}^{3} + mu\frac{du}{d\theta}^{2}\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{L^2}{m}u^6\frac{du}{d\theta}\frac{d\theta}{dt} - GMmu^5\frac{du}{d\theta}\frac{d\theta}{dt} = 0$$