Conservation of Energy

$$T + U = constant$$

$$\frac{m(\dot{x}^2 + \dot{y}^2)}{2} - \frac{GMm}{r} = constant$$

$$x = r\,cos(\theta), y = r\sin(\theta)$$

$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta} \text{ (1)}$$

$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta} \text{ (2)}$$

$$m\frac{\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right)^2 + \left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right)^2}{2} - \frac{GMm}{r} = 0$$

$$\tfrac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\theta}^2\right) - \tfrac{GMm}{r} = constant \text{ (1)}$$

Conservation of Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p} = constant$$

$$r = \hat{\imath}\, r\cos(\theta) + \hat{\jmath}\, r\sin(\theta) + \hat{k}\, 0$$

Using (1) and (2)

$$p = \hat{\imath}\, m\dot{x} + \hat{\jmath} m\dot{y} = \hat{\imath}\, m\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right) + \hat{\jmath} m\left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right)$$

$$\vec{L} = \vec{r} \times \vec{p} = \hat{k}\, [r\cos(\theta)\;\; m\left(\dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}\right) - \;\; r\sin(\theta)\;\; m\left(\dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}\right)]$$

Simplifying yields

$$\vec{L} = mr^2\dot{\theta} = constant$$

$$\dot{\theta} = \frac{L}{mr^2}$$

Substituting this in (1)

$$\frac{1}{2}\left(m\dot{r}^2 + mr^2\frac{L^2}{m^2r^4}\right) - \frac{GMm}{r} = constant$$

$$\tfrac{1}{2}\left(m\dot{r}^2 + \tfrac{L^2}{mr^2}\right) - \tfrac{GMm}{r} = constant \text{ (3)}$$

Using $h = \frac{L}{m}$

$$\frac{1}{2}\left(m\dot{r}^2 + m\frac{h^2}{r^2}\right) - \frac{GMm}{r} = constant$$

Substitute $r = \frac{1}{u(\theta)}$

$$\dot{r} = -\frac{1}{u^2}u'\dot{\theta}$$

$$\frac{1}{2}\left(\frac{m}{u^4}u'^2\dot{\theta}^2 + mh^2u^2\right) - GMmu = constant$$

Using $\dot{\theta} = \frac{h}{r^2}$

$$\frac{1}{2}\left(\frac{m}{u^4}u'^2\frac{h^2}{r^4} + mh^2u^2\right) - GMmu = constant$$

$$\frac{1}{2}\left(mu'^2h^2 + mh^2u^2\right) - GMmu = constant$$

Differentiating with respect to time

$$\frac{1}{2}\left(m2u'u''\dot{\theta}h^2 + mh^2 2uu'\dot{\theta}\right) - GMmu'\dot{\theta} = 0$$

$$u'' + u - \frac{GM}{h^2} = 0$$

$$u = Acos\theta + Bsin\theta + \frac{GM}{h^2}$$

Derivative of Energy Expression with respect to time

$$\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta} + \frac{GM}{r^2}\dot{r} = 0$$

Derivative of Angular Momentum Expression with respect to time

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}$$

$$\dot{r}\ddot{r} + r\dot{r}\dot{\theta}^2 - r^2\dot{\theta}\frac{2\dot{r}\dot{\theta}}{r} + \frac{GM}{r^2}\dot{r} = 0$$

$$\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r^2} = 0$$

Using $\dot{\theta} = \frac{h}{r^2}$

$$\ddot{r} - \frac{h^2}{r^3} + \frac{GM}{r^2} = 0$$