Exercise 9.5.

$$\sqrt{\frac{\pi}{2z}} I_{n+1/2}(z) \xrightarrow[z\to\infty]{} \sim \frac{\mathrm{e}^{z}}{z} \quad ,$$

$$\sqrt{\frac{\pi}{2z}} K_{n+1/2}(z) \xrightarrow[z\to\infty]{} \sim \frac{\mathrm{e}^{-z}}{z} \quad . \tag{20}$$

Before continuing, we want to give the representation of the $\chi_{\kappa,\mu}(\vartheta,\varphi)$ in (11) [see Example 9.3 and Chap. 10, (10.32)]:

$$\chi_{\kappa,\mu}(\vartheta,\varphi) = \sum_{m=-1/2,1/2} \left(l_\kappa \tfrac{1}{2} j \middle| \mu - m, m\right) Y_{l_\kappa,\mu-m}(\vartheta,\varphi)\chi_{\frac{1}{2},m} \tag{21}$$

with

$$\chi_{\frac{1}{2},\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \chi_{\frac{1}{2},-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \tag{22}$$

Let us now find the bound states.[6] For these, $E > V_0 + m_0c^2$ and $-m_0c^2 < E < m_0c^2$. In the inner region of the potential field we must therefore take the solutions (15) and set $a_2 = 0$, in order that the wave functions remain normalizable at the origin. On the other hand in the outer region we must set $b_2 = 0$ in (17), so that the wave functions are normalizable at infinity. Both solutions must be joined at $r = R_0$. One can eliminate the normalizing constants a_1, b_1 by adjustment of the ratio u_1/u_2 at $r = R_0$. This gives

$$\frac{\kappa}{|\kappa|} \frac{R_0 j_{l_\kappa}(kR_0)}{kR_0 j_{l_{-\kappa}}(kR_0)} \left(E + V_0 + m_0c^2\right) = -\frac{R_0 K_{l_\kappa+\frac{1}{2}}(KR_0)}{KR_0 K_{l_{-\kappa}+\frac{1}{2}}(KR_0)} \left(E + m_0c^2\right) \tag{23}$$

and

$$\frac{j_{l_\kappa}(kR_0)}{j_{l_{-\kappa}}(kR_0)} = -\frac{\kappa}{|\kappa|}\frac{k}{K}\frac{K_{l_\kappa+\frac{1}{2}}(KR_0)}{K_{l_{-\kappa}+\frac{1}{2}}(KR_0)}\frac{E + m_0c^2}{E + V_0 + m_0c^2} \tag{24}$$

with

$$\hbar ck = \sqrt{(E + V_0)^2 - m_0^2c^4} \quad ,$$

$$\hbar cK = \sqrt{m_0^2c^4 - E^2} \quad . \tag{25}$$

For $|\kappa| = 1$ one can further simplify the equations analytically, and for s states ($\kappa = -1$, $l_\kappa = 0$, $l_{-\kappa} = 1$) this results in

$$\frac{kR_0 \sin kR_0}{\sin kR_0 - kR_0 \cos kR_0} = +\frac{k}{K}\frac{\mathrm{e}^{-KR_0}}{\mathrm{e}^{-KR_0}(1 + 1/KR_0)}\frac{E + m_0c^2}{E + V_0 + m_0c^2} \quad . \tag{26}$$

After some transformations one gets

$$\tan\left[\frac{R_0}{\hbar c}\sqrt{(E + V_0)^2 - m_0^2c^4}\right]\sqrt{\frac{E + V_0 + m_0c^2}{E + V_0 - m_0c^2}}$$

$$\times\left\{\frac{\hbar c}{R_0}\left[\frac{1}{E + V_0 + m_0c^2} - \frac{1}{E + m_0c^2}\right] - \sqrt{\frac{m_0c^2 - E}{m_0c^2 + E}}\right\} = 1 \quad . \tag{27}$$

[6] W. Pieper, W. Greiner: Z. Phys. **218**, 327 (1969).

Analogously one obtains for $p_{1/2}$ states ($\kappa = 1$, $l_\kappa = 1$, $l_{-\kappa} = 0$): *Exercise 9.5.*

$$\frac{kR_0 \sin kR_0}{\sin kR_0 - kR_0 \cos kR_0} = -\frac{K}{k}\frac{1}{1 + 1/KR_0}\frac{E + V_0 + m_0c^2}{E + m_0c^2} \tag{28}$$

and

$$\tan\left[\frac{R_0}{\hbar c}\sqrt{(E + V_0)^2 - m_0^2c^4}\right]\sqrt{\frac{E + V_0 - m_0c^2}{E + V_0 + m_0c^2}} \times \left\{\frac{\hbar c}{R_0}\left[\frac{1}{E + V_0 - m_0c^2} + \frac{1}{m_0c^2 - E}\right] + \sqrt{\frac{m_0c^2 + E}{m_0c^2 - E}}\right\} = 1 \quad . \tag{29}$$

Another form for (26) is (defining $\alpha = kR_0$)

$$\alpha \cot \alpha - 1 = \frac{(m_0c^2/\hbar c)R_0\sqrt{1 - (E^2/m_0^2c^4)} + 1}{1 + E/m_0c^2} \times \left(1 + \sqrt{1 + \left(\frac{a\hbar c}{m_0c^2R_0}\right)^2}\right) \quad , \tag{30}$$

and for (28) one can write:

$$1 - \alpha \cot \alpha = \frac{(m_0c^2/\hbar c)R_0\sqrt{1 - (E^2/m_0^2c^4)} + 1}{1 - E/m_0c^2} \times \left(-1 + \sqrt{1 + \left(\frac{a\hbar c}{m_0c^2R_0}\right)^2}\right) \quad . \tag{31}$$

From (27) and (29) we can now calculate the energy eigenvalues of $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ states. If we assume R_0 to be small ($m_0cR_0/\hbar \ll 1$), we may solve (27) and (29) approximately by expanding in terms of $m_0cR_0/\hbar$. A short calculation (which is left as an exercise for the reader) leads to the following Table 9.1, where $n = 1, 2, 3, \ldots$ labels the states. Similarly, one can find approximate solutions for (30) and (31)

Table 9.1. Energy eigenvalues for $s_{1/2}$ and $p_{1/2}$ states in a small potential box

E	$V_0(\kappa = -1)$	$V_0(\kappa = +1)$
m_0c^2	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} - 3m_0c^2$	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} - m_0c^2$
0	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} - m_0c^2$	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} + m_0c^2$
$-m_0c^2$	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} + m_0c^2$	$\frac{\hbar}{m_0c}\frac{n\pi}{R_0} + 3m_0c^2$

$n = 1, 2, 3, \ldots$ $(m_0cR_0/\hbar) \ll 1$

for the opposite limiting case of a very large potential box ($m_0cR_0/\hbar \gg 1$), and this is shown in Table 9.2. One sees that the $p_{\frac{1}{2}}$ states are energetically higher than

Exercise 9.5.

Table 9.2. Energy eigenvalues for $s_{1/2}$ and $p_{1/2}$ states in a large potential box

E	$V_0(\kappa = -1)$	$V_0(\kappa = +1)$
m_0c^2	$\frac{(n-1/2)^2\pi^2\hbar^2}{2m_0R_0^2}$	$\frac{n^2\pi^2\hbar^2}{2m_0R_0^2}$
0	$m_0c^2\left(1+\frac{n^2\pi^2\hbar^2}{2m_0^2c^2R_0^2}\right)$	$m_0c^2\left(1+\frac{(n+1/2)^2\pi^2\hbar^2}{2m_0^2c^2R_0^2}\right)$
$-m_0c^2$	$2m_0c^2\left(1+\frac{n^2\pi^2\hbar^2}{4m_0^2c^2R_0^2}\right)$	$2m_0c^2\left(1+\frac{(n+1/2)^2\pi^2\hbar^2}{4m_0^2c^2R_0^2}\right)$

$n = 1, 2, 3, \ldots$ $(m_0cR_0/\hbar) \gg 1$

the s states, which can be understood intuitively because of the orbital angular momentum $l = 1$ for the p states. But even for the s states (with $l = 0$), for a given R_0 a minimal potential depth V_0 is required in order to get at least one bound state, in contrast to the one-dimensional problem, where at least one bound state always exists. This is due to the fact that for the s state of a Dirac particle in a three-dimensional potential well there is an angular momentum barrier due to the spin. Indeed, this can be easily seen by decoupling (14), differentiating again with constant V_0 and reinserting:

$$g'' - \left\{\frac{1}{(\hbar c)^2}\left[(E+V_0)^2 - m_0^2c^4\right] - \frac{\kappa(\kappa+1)}{r^2}\right\} g = 0 \quad ,$$
$$f'' + \left\{\frac{1}{(\hbar c)^2}\left[(E+V_0)^2 - m_0^2c^4\right] - \frac{\kappa(\kappa-1)}{r^2}\right\} f = 0 \quad . \tag{32}$$

On the one hand, for s states ($\kappa = -1$) the angular momentum barrier is zero for the large components. On the other, the equation for f contains an angular momentum term, which increases the energy in the three-dimensional case even for s states.

In Fig. 9.11 the eigenvalues[7] [found numerically from (27)] for the $1s$ state in potential wells with different values of R_0 have been plotted. One sees that for $(m_0cR_0/\hbar) \ll 1$ as well as for $(m_0cR_0/\hbar) \gg 1$ the energy eigenvalue $E(V_0)$ grows almost linearly with V_0. As in the one-dimensional case, we can determine the scattering phase shifts of the continuum. For the s waves this can be done with little effort, whereas for the waves with higher angular momentum the matching condition at $r = R_0$ cannot be evaluated easily. Let us therefore look at the scattering phase shifts of the s waves. First we have to match solutions of the interior region,

$$u_1^{\mathrm{i}}(r) = a_1 \sin k_{\mathrm{i}} r \quad ,$$
$$u_2^{\mathrm{i}}(r) = -a_1\sqrt{\frac{E+V_0-m_0c^2}{E+V_0+m_0c^2}}\left(\frac{\sin k_{\mathrm{i}}r}{k_{\mathrm{i}}r} - \cos k_{\mathrm{i}}r\right) \quad , \tag{33}$$

at $r = R_0$ to the solution $\left((\hbar c k_{\mathrm{o}})^2 = E^2 - m_0^2c^4\right)$ of the outside region:

[7] From J. Rafelski, L. Fulcher, A. Klein: Phys. Rep. **38**, 227 (1978).