

Notice that $P_l(x)$ is (as the name suggests) an l th-order *polynomial* in x ; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front ($1/2^l l!$) was chosen in order that

$$P_l(1) = 1. \quad (3.63)$$

The Rodrigues formula obviously works only for nonnegative integer values of l . Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second-order*, and it should possess *two* independent solutions, for *every* value of l . It turns out that these “other solutions” blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.¹³ For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln \left(\tan \frac{\theta}{2} \right). \quad (3.64)$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general separable solution to Laplace’s equation, consistent with minimal physical requirements, is

$$V(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}} \right) P_l(\cos \theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l . The *general* solution is the linear combination of separable solutions:

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta). \quad (3.65)$$

The following examples illustrate the power of this important result.

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R . Find the potential inside the sphere.

Solution

In this case, $B_l = 0$ for all l —otherwise the potential would blow up at the origin. Thus,

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta). \quad (3.66)$$

¹³In rare cases where the z axis is excluded, these “other solutions” do have to be considered.

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta) = V_0(\theta). \quad (3.67)$$

Can this equation be satisfied, for an appropriate choice of coefficients A_l ? Yes: The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \leq x \leq 1$ ($0 \leq \theta \leq \pi$). How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are *orthogonal* functions:¹⁴

$$\begin{aligned} \int_{-1}^1 P_l(x) P_{l'}(x) dx &= \int_0^\pi P_l(\cos \theta) P_{l'}(\cos \theta) \sin \theta d\theta \\ &= \begin{cases} 0, & \text{if } l' \neq l, \\ \frac{2}{2l+1}, & \text{if } l' = l. \end{cases} \end{aligned} \quad (3.68)$$

Thus, multiplying Eq. 3.67 by $P_{l'}(\cos \theta) \sin \theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^\pi V_0(\theta) P_{l'}(\cos \theta) \sin \theta d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta. \quad (3.69)$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69.

It can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 “by eyeball.”¹⁵ For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = k \sin^2(\theta/2), \quad (3.70)$$

where k is a constant. Using the half-angle formula, we rewrite this as

$$V_0(\theta) = \frac{k}{2} (1 - \cos \theta) = \frac{k}{2} [P_0(\cos \theta) - P_1(\cos \theta)].$$

¹⁴M. Boas, *Mathematical Methods in the Physical Sciences*, 2nd ed. (New York: John Wiley, 1983), Section 12.7.

¹⁵This is certainly true whenever $V_0(\theta)$ can be expressed as a polynomial in $\cos \theta$. The degree of the polynomial tells us the highest l we require, and the leading coefficient determines the corresponding A_l . Subtracting off $A_l R^l P_l(\cos \theta)$ and repeating the process, we systematically work our way down to A_0 . Notice that if V_0 is an *even* function of $\cos \theta$, then only even terms will occur in the sum (and likewise for odd functions).

Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other A_l 's vanish. Therefore,

$$V(r, \theta) = \frac{k}{2} \left[r^0 P_0(\cos \theta) - \frac{r^1}{R} P_1(\cos \theta) \right] = \frac{k}{2} \left(1 - \frac{r}{R} \cos \theta \right). \quad (3.71)$$

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R , but this time we are asked to find the potential *outside*, assuming there is no charge there.

Solution

In this case it's the A_l 's that must be zero (or else V would not go to zero at ∞), so

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta). \quad (3.72)$$

At the surface of the sphere, we require that

$$V(R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) = V_0(\theta).$$

Multiplying by $P_{l'}(\cos \theta) \sin \theta$ and integrating—exploiting, again, the orthogonality relation 3.68—we have

$$\frac{B_{l'}}{R^{l'+1}} \frac{2}{2l'+1} = \int_0^\pi V_0(\theta) P_{l'}(\cos \theta) \sin \theta \, d\theta,$$

or

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta \, d\theta. \quad (3.73)$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Example 3.8. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{z}}$. The field will push positive charge to the “northern” surface of the sphere, and—symmetrically—negative charge to the “southern” surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.