

$$H = \sum_{i,j} a_i^\dagger h_{ij} a_j$$

H is diagonalized by the standard procedure and using that U is unitary:

$$UHU^{-1} = UHU^\dagger$$

Or in terms of indices:

$$(UHU^\dagger)_{ij} = \sum_{l,k} U_{ik} H_{kl} U_{lj}^\dagger = \delta_{ij} \varepsilon_i$$

The last equality comes from demanding that the new matrix be diagonalized. But I need the matrix elements of H in the old basis and I don't know what the old basis even is?