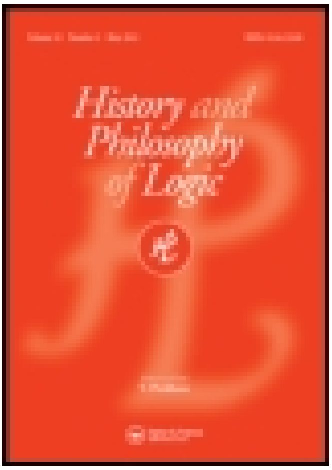

History and Philosophy of Logic
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/thpl20

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Published online: 25 Sep 2014.

To cite this article: John Corcoran & Hassan Masoud (2014): Existential Import Today: New Metatheorems; Historical, Philosophical, and Pedagogical Misconceptions, History and Philosophy of Logic, DOI: 10.1080/01445340.2014.952947

To link to this article: http://dx.doi.org/10.1080/01445340.2014.952947

HISTORY AND PHILOSOPHY OF LOGIC, 2014
http://dx.doi.org/10.1080/01445340.2014.952947

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Existential Import Today: New Metatheorems; Historical, Philosophical, and Pedagogical Misconceptions

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Received 24 June 2014 Accepted 5 July 2014

Contrary to common misconceptions, today's logic is *not* devoid of existential import: the universalized *conditional* $\forall x\,[\mathrm{S}(x) \rightarrow \mathrm{P}(x)]$ implies its corresponding existentialized *conjunction* $\exists x\,[\mathrm{S}(x)\ \&\ \mathrm{P}(x)]$, not in all cases, *but in some*. We characterize the proexamples by proving the *Existential-Import Equivalence*:

$$\forall x\,[\mathrm{S}(x) \rightarrow \mathrm{P}(x)] \text{ implies } \exists x\,[\mathrm{S}(x)\&\ \mathrm{P}(x)] \text{ iff } \exists x\ \mathrm{S}(x) \text{is logically true.}$$

The antecedent S(x) of the universalized conditional alone determines whether the universalized conditional *has existential import*, i.e. whether it implies its corresponding existentialized conjunction.

A *predicate* is an open formula having only x free. An *existential-import* predicate Q(x) is one whose existentialization, $\exists x$ Q(x), is logically true; otherwise, Q(x) is *existential-import-free* or simply *import-free*.

How abundant or widespread is existential import? How abundant or widespread are existential-import predicates in themselves or in comparison to import-free predicates? We show that existential-import predicates are quite abundant, and no less so than import-free predicates. Existential-import implications hold as widely as they fail. Existential import is not an isolated phenomenon. As documented below, these results correct false or misleading passages even in respected logic texts.

1. Introduction

> A particular conclusion cannot be validly drawn from a universal premise, or from any number of universal premises. (*Lewis and Langford 1932*, p. 62)

The mathematical results in this paper are not difficult; they could have been discovered and proved as early as the 1920s. Nevertheless, even though they clarify central aspects of standard first-order logic, they remained hidden for over 80 years. As is often the case, the difficulties were more with discovery of the theorems than with discovery of their proofs — given what had been contributed by previous logicians. One reason they remained hidden might be that they concern an aspect of the established paradigm that invites tedious comparison with the outmoded paradigm — a comparison that confuses beginners and that promises no reward to experts. Let us begin.

The universalized conditional $\forall x(x = 0 \rightarrow x = (x + x))$ implies the corresponding existentialized conjunction $\exists x(x = 0\ \&\ x = (x + x))$. And $\exists x(x = 0)$ is *tautological* (in the broad sense, i.e. logically true).

But $\forall x(x = (x + x) \rightarrow x = 0)$ does not imply $\exists x(x = (x + x)\ \&\ x = 0)$. And $\exists x(x = (x + x))$ is *informative* (non-tautological). The Existential-Import Equivalence says that these examples are typical.

In the first example, which apparently contradicts many logic books including *Lewis and Langford's 1932* *Symbolic Logic* quoted above, the 'particular [sc. existential] conclusion' is informative.[1] However, we should not overlook the other exceptions to the quoted assertion that have tautological 'particular conclusions': for example, the corresponding existentialized conjunction of the tautology $\forall x\,(x = x \rightarrow x = x)$ is likewise tautological. Many logic texts — some otherwise highly competent such as *Goldfarb 2003* (p. 108) — contain misconceptions and errors resulting from insufficient attention to the facts just stated. See Section 6 for details.

The 'rule' that no existentialized conjunction is implied by the corresponding universalized conditional has many exceptions, many of which are trivial but many of which are far from trivial — as we will see.

Implication and tautology: As usual, a sentence or set of sentences *implies*, or *is an implicant of*, a given sentence iff the given sentence is satisfied by every interpretation satisfying the sentence or set of sentences. Conversely, a given sentence *follows from*, *is a consequence of*, *is implied by*, or *is an implication of* a sentence or set of sentences iff the sentence or set of sentences implies the given sentence.[2]

One given sentence *is logically equivalent to* or *is a logical equivalent of* a second iff each implies the other. One given set of sentences *is logically equivalent to* or *is a logical equivalent of* a second set iff each set implies every member of the other. As usual, we occasionally use expressions such as 'the sentence $0 = 1$ implies [...]' elliptically for corresponding expressions 'the unit set (or singleton) of the sentence $0 = 1$ implies [...]'.[3]

Sentences that are logically equivalent to the null set are said — quite naturally — to be *uninformative*, or *tautological* (logically true), or to be *tautologies* (logical truths).[4] Tautological sentences are thus satisfied by every interpretation. Sentences that are non-tautological are said to be *informative*.

It will be useful in this paper to notice that in order to show that a given sentence is tautological it is sufficient to show that it is an implication of the negation of one of its own implicants — in other words that it is implied by some sentence whose negation also implies it.[5]

The 'truth-functional tautologies', which can be seen to be tautological in virtue of their truth-functional connectives, play little or no role in this paper. Most of the tautologies of interest here are existential sentences: in fact, existentializations of predicates that do not have truth-functional tautologies as instances.[6] Where t is an individual constant or

1 Notice that the existentialized conjunction $\exists x(x = 0\,\&\,x = (x + x))$ in the first example is in the same logical form as $\exists x(x = 1\,\&\,x = (x + x))$, which is false under the usual interpretation. For logical form, see Section 5.

2 See, e.g. *Goldfarb 2003* (pp. 129f and 175f), or *Boolos et al. 2007* (pp. 101f and 119f). This is the now standard 'no countermodels conception of consequence' often mistakenly attributed to Alfred Tarski's paper 'On the concept of logical consequence' (*1936*, pp. 409–16). For details, see *Corcoran and Sagüillo 2011*.

3 This simply avoids awkward and unnatural repetition of the four words 'the unit set of'. It is not taking license for 'abuse of language' and it is certainly not 'identifying a sentence with its singleton, or unit set'. There are people — such as *Quine 1987* — who recommend such 'identification' but we are not among them.

4 Thus, the tautologies are not exhausted by the so-called truth-functional tautologies. See the following text, and also see *Audi 1999* (p. 902f).

5 Since every sentence is one of its own implicants, a special case is showing that a sentence is tautological by showing that it is an implication of its own negation. This special case does not arise in this paper. Being an implication of its own negation is a necessary but non-sufficient condition for being the negation of one of its own implications. $0 = 0$ is an implication of its own negation but it is not the negation of anything. Of course, by the negation of a sentence is meant the result of prefixing it with '$\sim$', the negation sign. Thus, no sentence is its own negation.

6 By an *instance of a predicate* $Q(x)$ is meant the result of substituting a constant term for all free occurrences of its variable x. Allowing for differences of conceptual framework, this usage agrees remarkably well with that of 'instance of a propositional function' in Lecture II of *Russell 1914*. The word 'instance' has many other standard uses in logic (*Church 1956*, p. 13,

a constant term such as a numeral, the sentence $\exists x\ (x = t)$ is tautological and none of the instances of its predicate $(x = t)$ is tautological — except $(t = t)$, which is not truth-functionally tautological. Many other examples are given below.

The use of the expressions 'tautological sentence' and 'tautology' in a broad sense also expressed by 'logical truth', 'logical proposition', 'analytic truth', 'truth of logic', 'valid sentence', and others goes back at least as far as *Russell 1919* (pp. 194–206).[7] A little later, in the 1930s, influential papers by Gödel and others also recognize the use of 'tautology' in a broad sense, perhaps following Russell's usage to some extent.[8]

Although the conventions used here are relatively standard, there are other equally acceptable conventions for using the word 'implies' and the other terms just introduced (*Corcoran 1973*, *Goldfarb 2003*). Of course, historical conventions are hardly relevant to modern usage. For example, early writers used 'logical truth' for informative metalinguistic propositions such as the laws of contradiction and excluded middle as well as for conditionals whose antecedents imply their consequents (*De Morgan 1847*, p. 1).

Scope and limits: This essay concerns standard one-sorted, first-order logics *in themselves* — without regard to their applications or to how their languages are used to translate normal or mathematical English.

One-sorted, first-order logics include the underlying logics — in the sense of *Church 1956* — of elementary number theory, set theory, geometry, and other sciences; they are not limited to monadic logics.[9] Here the expression 'first-order' indicates that in these logics all variables are individual variables. The expression 'one-sorted' indicates that in any given interpretation all variables range over one universe of discourse. The range of the variables is *the* universe of discourse.[10]

Recall that the standard one-sorted, first-order logics — often conveniently but misleadingly called first-order logics or predicate logics — require that in every interpretation the universe be non-empty and all non-logical constants have appropriate denotations: individual constants denote individuals, n-place relation constants denote n-place relations, sentence constants denote truth-values, etc.[11]

We do not consider *non-standard* logics such as many-valued, paraconsistent, intuitionistic, or presupposition-free logics nor do we treat other *standard* logics such as many-sorted, equational, or second-order logics. Below, 'first-order' abbreviates 'standard, one-sorted, first-order'.

Further, this essay has no interest in such challenging questions as whether the propositions normally expressed by the English sentences 'Every sphere is a polygon' and 'Some sphere is a polygon' — containing two nouns and no adjectives — are equally

Corcoran 2006, p. 219 passim; *Goldfarb 2003*, p. 57). In Section 5, we speak of instances of quantified sentences — not of predicates. This corresponds to Russell's 'instance of a proposition' (Lecture II of *Russell 1914*).

7 On page 203, Russell objects to saying that tautologies are propositions whose 'contradictories are self-contradictory' — but not on the ground that it would be false to say so. His objection seems to be that use of the condition 'having a self-contradictory contradictory' is misleading because it might seem to attribute to the 'law of contradiction' a role that is also played by many other 'logical propositions'. Maybe he would not object to saying that tautologies are propositions that are implied by their own negations.

8 See, for example, *Audi 1999* (p. 903) and the widely read *Lewis and Langford 1932* (p. 24, passim).

9 The expression 'monadic logics' is used in the modern sense in which they are first-order logics whose predicates are all monadic [sc. involve only one variable] as in *Goldfarb 2003* (p. 91ff) — as opposed to the original sense where they are devoid of quantifiers as in *Hilbert and Ackermann 1928/1938/1950* (p. 44ff).

10 To make the same point, some logicians prefer to say 'variables range over the objects in the universe of discourse' or that the universe is 'the range of values of the variables' (*Quine 1970/1986*, p. 25, 52). Others prefer to say 'The range of the quantifiers is the universe of discourse' (*Goldfarb 2003*, p. 119). For other terminology and alternative viewpoints, see *Corcoran 1999*.

11 In this essay, no use is made of sentence constants although there is no mathematical reason to exclude them.

well-expressed or even better-expressed using instead ‘Every individual that is spherical is polygonal’ and ‘Some individual that is spherical is polygonal’ — containing one noun and two adjectives.[12]

Likewise beyond the scope of this essay are questions such as whether a statement made using a universal sentence such as ‘Every sphere is a polygon’ might carry conversational implicatures not logically implied by the proposition stated (*Corcoran 2009b*). And of course, the *pragmatic* or *conversational* phenomenon of ‘enthymemic implication’ is totally beside the point here: whenever we say one given proposition implies another given proposition, we mean that the former *by itself* implies the latter.[13] However, in Section 5, one small, purely *semantic* point superficially related to pragmatic ‘enthymemic implication’ will be treated to suggest possible applications to philosophy of logic and to identify misconceptions in the literature.

Furthermore, although some historical remarks appear in Section 5, they are all about the history of presentations of first-order logic. There is no concern with the history of existential import in the context of ‘traditional logic’: syllogistic extended with ‘negative terms’.[14]

Finally, this essay is not applying modern mathematical logic to Aristotle’s logic as in *Corcoran 1972*. It has no interest in the question of whether there is agreement or disagreement between Aristotle’s logic and any one or more of the modern logics that have been developed since Boole’s 1847 logic. Moreover, there is likewise no interest in the question, assuming disagreement, of which logic if any is ‘right’. As far as this essay is concerned, different logics may well be ‘incommensurable’ in Kuhn’s sense[15] and logical theories may well be ‘unfalsifiable’ in the sense of Popper.

With these essential preliminaries settled, let us turn to the subject of the paper: ‘existential import’.

Existential import: Aristotle’s logic has *unlimited* existential import: the universal affirmative ‘P *belongs-to-every* S’ implies the corresponding existential affirmative ‘P *belongs-to-some* S’ in *every* case. In other words, also in keeping with Aristotle’s own terminological alternatives, the universal affirmative ‘*every* S *is a* P’ implies the corresponding existential affirmative ‘*some* S *is a* P’ in *every* case. In this respect, it is similar to modern many-sorted logic (*Corcoran 2008*):

$$\forall s \exists p (s = p) \text{ implies } \exists s \exists p (s = p).$$

Smiley 1962, *Parry 1966*, and others showed how Aristotle’s categorical syllogistic can be faithfully represented in modern symbolic logic — if many-sorted first-order logic is used instead of the usual one-sorted first-order logic.[16] In many-sorted logic, each sort of variable suggests a common noun.

12 See *Church 1956* (pp. 4, 269, and 318) and *Gupta 1980* (passim).

13 *Church 1965* (p. 419) ties this point to what he calls the formality of logic. He wrote:

> Strawson’s position betrays a lamentable lack of concern for the formality of logic. Adequate formality requires that any matter of extralinguistic fact which must be known before an inference can be made shall be stated as a premiss of that inference.

14 This topic is treated succinctly and accurately in Church’s classic paper (*1965*).

15 Quine seemed to suggest this (*1970/1986*, p. 96).

16 Recently Neil Tennant, one of Smiley’s PhD students from the 1970s, made some interesting remarks related to these points in *2014*. This is a convenient place to point out that *Quine 1970/1986* (p. 25) is simply mistaken if he thinks that many-sorted logic is reducible to one-sorted. Otherwise, Aristotle’s logic would be reducible to monadic first-order.

Already in the 1950s, Church suggested that common nouns, such as ‘sphere’ and ‘polygon’, could correspond to variables in modern logic (*1956*, pp. 4 and 318) and that the ‘subject’ and ‘predicate terms’ of the traditional syllogism might be fruitfully construed as common nouns or as variables (*1956*, p. 269). Even before 1900, Hilbert had used in effect a three-sorted underlying logic for his geometry: uppercase Latin letters associated with the common noun ‘point’, lowercase Latin letters associated with the common noun ‘line’, and lowercase Greek letters associated with the common noun ‘plane’ (*Hilbert 1899/1971*, p. 2ff).[17]

As taught in many introductory symbolic logic courses, even in one-sorted first-order logic, every universal sentence implies the corresponding existential:

$$\forall x\, Q\,(x) \text{ implies } \exists x\, Q(x).$$ [18]

Thus, the universalized conditional sentence $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ implies the corresponding existentialized *conditional* $\exists x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ in *all* cases — a point whose omission leaves room for misconceptions. However, we are interested in the relations of universalized conditional sentences $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ to their respective corresponding existentialized *conjunctions* $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$.

In contrast to Aristotle’s logic, as noted in the abstract above, one-sorted first-order logic has *limited* existential import: the universalized conditional sentence $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ implies the corresponding existentialized conjunction $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ in *some but not all* cases. The *Existential-Import Equivalence* (*Corcoran 2007*, p. 144) determines which implications hold:

$$\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x)) \text{ implies } \exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x)) \text{ iff } \exists x\ \mathrm{S}(x) \text{ is tautological.}$$

To be clear, the Existential-Import Equivalence is the proposition that in order for a universalized conditional sentence $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ to imply the corresponding existentialized conjunction $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ it is necessary and sufficient for $\exists x\ \mathrm{S}(x)$, the existentialization of the antecedent predicate, to be tautological. Sufficiency, ‘if’, is obvious enough. The easily proved necessity, ‘only if’, can be established using reasoning similar to that used by *Corcoran and Masoud 2014*. A new proof is given in Section 2.

The necessary and sufficient condition is that $\exists x\ \mathrm{S}(x)$, the existentialization of the antecedent predicate, be tautological; it is not the condition that the corresponding existentialized conjunction $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ be tautological. The latter condition is merely sufficient.

Of course, if $\exists x\ \mathrm{S}(x)$ is tautological, then $\mathrm{S}(x)$ cannot be an atomic predicate composed of a predicate letter, say A, followed by a string of one or more occurrences of x. None of the following sentences is tautological: $\exists x\ \mathrm{A}x$, $\exists x\ \mathrm{A}xx$, $\exists x\ \mathrm{A}xxx$, and so on. This corresponds to facts such as that ‘Some person is invisible’ is not tautological. Of course, ‘Every person who is invisible is weightless’ does not imply ‘Some person is invisible and weightless’.

Notice that a consequence of the Existential-Import Equivalence is that whether an existential-import implication holds is independent of the form and content of the consequent $\mathrm{P}(x)$ — in the sense that if $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ then $\forall x\ (\mathrm{S}(x) \to \mathrm{Q}(x))$ implies $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{Q}(x))$ no matter which predicate $\mathrm{Q}(x)$ is used. Of course, it also follows that if $\forall x\ (\mathrm{S}(x) \to \mathrm{P}(x))$ does not imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{P}(x))$ then $\forall x\ (\mathrm{S}(x) \to \mathrm{Q}(x))$ does not imply $\exists x\ (\mathrm{S}(x)\ \&\ \mathrm{Q}(x))$ no matter which predicate $\mathrm{Q}(x)$ is used.

[17] See also *Gupta 1980* and *Corcoran 2008*, *2009a*.

[18] See, e.g., *Reichenbach 1947* (p. 93), *Suppes 1957/1999* (p. 68), *Lemmon 1965/1978* (pp. 112 and 176), and *Goldfarb 2003* (p. 144).

To discuss further consequences of the Existential-Import Equivalence, it is useful to officially recognize and expand terminology already used here and elsewhere.[19]

Existential-Import Terminology: Let us say that a given universalized conditional *has existential import* if it implies the corresponding existentialized conjunction. It may seem awkward at first but we will also say that a given existentialized conjunction *has existential import* if it is implied by the corresponding universalized conditional.

By an *implication*, we mean a metalogical proposition to the effect that a certain sentence implies another.[20] By an *existential-import implication*, we mean an implication to the effect that a certain universalized conditional implies the corresponding existentialized conjunction. In other words, an *existential-import implication* is a metalogical proposition such as

$$\forall x\,(\mathrm{S}(x) \to \mathrm{P}(x)) \text{ implies } \exists x\,(\mathrm{S}(x)\,\&\,\mathrm{P}(x)),$$

where $\mathrm{S}(x)$ and $\mathrm{P}(x)$ are specific predicates — possibly long and complex. Thus, the following is the existential-import implication, where $\mathrm{S}(x)$ is $x = 0$ and $\mathrm{P}(x)$ is $x = (x + x)$.

$$\forall x(x = 0 \to x = (x + x)) \text{ implies } \exists x(x = 0\,\&\,x = (x + x)).$$

The *premise* is $\forall x\,(\mathrm{S}(x) \to \mathrm{P}(x))$; the *conclusion* is $\exists x\,(\mathrm{S}(x)\,\&\,\mathrm{P}(x))$; the *determinant* is $\exists x\,\mathrm{S}(x)$; the *determinant predicate* is $\mathrm{S}(x)$; and the *second predicate* is $\mathrm{P}(x)$. Of course, in such contexts, 'holds' means 'is true'; 'fails' means 'is false'.[21]

In every existential-import implication, the premise $\forall x\,(\mathrm{S}(x) \to \mathrm{P}(x))$, the conclusion $\exists x\,(\mathrm{S}(x)\,\&\,\mathrm{P}(x))$, and the determinant $\exists x\,\mathrm{S}(x)$ are all sentences, 'closed' sentences having no free occurrences of variables — they are not predicates. Notice that every universalized conditional is the premise of one and only one existential-import implication and that every existentialized conjunction is the conclusion of one and only one existential-import implication. Of course, every universalized conditional has one and only one corresponding existentialized conjunction. Conversely, every existentialized conjunction is the corresponding existentialized conjunction of one and only one universalized conditional. This justifies attributing existential import to existentialized conjunctions as well as to universalized conditionals.

Using this terminology, we can say that a given universalized conditional has existential import if it is the premise of a true existential-import implication and that a given existentialized conjunction has existential import if it is the conclusion of a true existential-import implication.

The Existential-Import Equivalence says that in order for an existential-import implication to hold it is necessary and sufficient for the determinant to be tautological. The fact

[19] For the record, we note that our use of the expression 'existential import' does not always reflect its uses by others in the 130-odd years since its coinage by *Keynes 1884/1906* (pp. xv, 210ff). Concern with existential import predates its 'christening': the great medieval logician William of Ockham (c. 1285–1349) recognized empty terms and constructed a semantics for categorical sentences to preserve all of Aristotle's syllogisms (*Corcoran 1981*). However, as late as 1847, otherwise competent logicians such as Boole and De Morgan saw nothing remarkable about existential import in Aristotle's system: the first example in *De Morgan 1847* derives a particular from two universals.

[20] This natural use of the pluralizable common noun 'implication' is widespread today. It occurs dozens of times in *Goldfarb 2003* (pp. 53, 131, 135f, 145, 182) and in *Boolos et al. 2007* (p. 189, passim). Note that 'an implication' does not here mean 'a sentence that is an implication of some set of sentences' — a natural but useless sense: every sentence is an implication in that sense. To be clear, an implication is not a conditional sentence (*Corcoran 1973*). Just as 'an implication' is used for a proposition to the effect that one sentence implies another, 'an identity', 'an equality', 'an equivalence', 'an inequality', etc. are used for propositions to the effect that one object bears the indicated relation to another. The sad practice of calling conditionals *implications* is a vestige of the period before the meanings of 'implies' had been sorted out. See *Corcoran 1973*.

[21] This use of 'implication' as a pluralizable common noun denoting metalinguistic propositions is similar to that in *Goldfarb 2003* (pp. 51, 141, 181f, and 282). In this use, 'implication' does not denote a relation nor is it part of a relational verb as 'is an implication of'.

that the truth or falsity of an existential-import implication is independent of the second predicate means that in determining the truth-value of an existential-import implication it is never necessary to consider the non-logical constants not in the determinant. To show that an existential-import implication is true it is sufficient to deduce the determinant either from a known tautology or from its own negation or from the negation of one of its known implicants. To show that an existential-import implication is false it is sufficient to produce an interpretation that falsifies the determinant and this need not consider any non-logical constants not in the determinant. These observations greatly simplify the study of existential-import implications.

For example, to show that $\forall x[x = 1 \rightarrow \sim\exists y\, x = (y + 0)]$ *has* existential import it is sufficient to deduce the determinant $\exists x(x = 1)$ from $(1 = 1)$.

Also, to show that $\forall x[\sim(x = 1) \rightarrow \exists y\, x = (y + 0)]$ *does not have* existential import it is sufficient to notice that the determinant $\exists x\sim(x = 1)$ is false under any singleton interpretation.

It is useful to notice that in many cases, when a given universalized conditional $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ has existential import, its logical equivalents that are universalized conditionals do not all have existential import. In particular, there are cases where $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ has existential import but its contrapositive $\forall x\ (\sim\mathrm{Q}(x) \rightarrow \sim\mathrm{S}(x))$ does not. Examples are given below. Perhaps the most extreme examples involve tautological antecedents and consequents. An example given above is $\forall x\ (x = x \rightarrow x = x)$, whose corresponding existentialized conjunction is tautological. The contrapositive is $\forall x\ (\sim(x = x) \rightarrow \sim(x = x))$, also tautological of course. But $\exists x\ (\sim(x = x)\ \&\sim(x = x))$, the existentialized conjunction corresponding to the contrapositive, is not only non-tautological; it is self-contradictory.[22]

By the contrapositive of a universalized conditional $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ is meant $\forall x\ (\sim\mathrm{Q}(x) \rightarrow \sim\mathrm{S}(x))$, the universalization of the conditional's contrapositive. And by the contrapositive of a conditional predicate $(\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ is meant $(\sim\mathrm{Q}(x) \rightarrow \sim\mathrm{S}(x))$, the result of negating the components of the conditional's converse.[23]

To be sure, there are cases where $\forall x\ (\mathrm{S}(x) \rightarrow \mathrm{Q}(x))$ has existential import and its contrapositive $\forall x\ (\sim\mathrm{Q}(x) \rightarrow \sim\mathrm{S}(x))$ does too. Examples are given in the next paragraph. Perhaps the most extreme examples involve tautological antecedents and self-contradictory consequents as $\forall x\ (x = x \rightarrow \sim(x = x))$, whose corresponding existentialized conjunction is self-contradictory. The contrapositive is $\forall x\ (\sim\sim(x = x) \rightarrow \sim(x = x))$, also self-contradictory, of course. But $\exists x\ (\sim\sim(x = x)\ \&\sim(x = x))$, the existentialized conjunction corresponding to the contrapositive, is also self-contradictory.

There are also examples that are familiar sentences of arithmetic. Consider the sentence $\forall x\ (x = 0 \rightarrow \sim(x = 1))$, which has existential import and whose contrapositive $\forall x\ (\sim\sim(x = 1) \rightarrow \sim(x = 0))$ has it too. Consider the original sentence and an equivalent of its contrapositive $\forall x\ (x = 1 \rightarrow \sim(x = 0))$ obtained by deleting the double negation.

[22] A sentence is *self-contradictory* iff it implies its own negation, in other words, iff it contradicts itself. Hilbert and others define the contradictory of a sentence to be what we call its negation (*Hilbert and Ackermann 1928/1938/1950*, p. 3ff). It would be natural to take a self-contradictory sentence to be one that is its own contradictory. We saw above that no sentence is its own contradictory in that sense. However, as is common, we define *a contradictory of* a given sentence as a sentence logically equivalent to the negation of the given sentence. Since, under every interpretation, a sentence and its negation have different truth-values, no sentence is one of its own contradictories in this sense either.

[23] In the following text, 'contrapositive' is used in a broader sense in which *a contrapositive* of a conditional is obtained by reversing the antecedent and consequent and then in each changing (adding or deleting) one negation sign. Thus $[\sim\mathrm{Q}(x) \rightarrow \sim\mathrm{S}(x)]$ has four contrapositives $[\mathrm{S}(x) \rightarrow \mathrm{Q}(x)]$, $[\sim\sim\mathrm{S}(x) \rightarrow \mathrm{Q}(x)]$, $[\mathrm{S}(x) \rightarrow \sim\sim\mathrm{Q}(x)]$, and $[\sim\sim\mathrm{S}(x) \rightarrow \sim\sim\mathrm{Q}(x)]$. But $[\sim\mathrm{Q}(x) \rightarrow \mathrm{S}(x)]$ has only two contrapositives $[\sim\mathrm{S}(x) \rightarrow \mathrm{Q}(x)]$ and $[\sim\mathrm{S}(x) \rightarrow \sim\sim\mathrm{Q}(x)]$. While $[\mathrm{Q}(x) \rightarrow \mathrm{S}(x)]$ has only one. For other usage, see *Church 1956* (pp. 102, 121, 146) and *Goldfarb 2003* (p. 26).

Notice that these two sentences are in the same logical form. It follows from the principle of form for implication that any two sentences in the same logical form both have existential import or both lack it (*Corcoran 2004*).[24]

Notice that the Existential-Import Equivalence somewhat justifies the terminology introduced in the abstract: a one-place predicate ('open formula' having only x free) Q(x) is *import-carrying* iff $\exists x$ Q(x) is tautological. An existential-import predicate is one that is import carrying.[25] Q(x) is *import-free* iff $\exists x$ Q(x) is not tautological.

Predicates: The word 'predicate' has been used in dozens, if not scores, of senses in the history of logic starting with the first book of Aristotle's *Organon*, whose title *Kategoriai* ('*Categories*') is often translated *Predicates*.[26] Our use agrees with that of several mathematicians and philosophers—but by no means all or even a majority. It agrees exactly or almost exactly with Michael Dummett's usage. In *Dummett 1973* (p. 10), he wrote the following.

> A sentence may be formed by combining a sign of generality with a one-place predicate. The one-place predicate is itself to be thought of as having been formed from a sentence by removing one or more occurrences of some one singular term (proper name).

It is clear from the context that Dummett intends the occurrences of the proper name to be replaced by occurrences of an individual variable that had not occurred in the original sentence.

This usage of 'predicate' is also very closely related to the Quine-Mates usage of 'English predicate'. According to *Mates 1972* (p. 77), '[. . .] an English predicate is like an English sentence except that it contains [. . .] counters [. . .] at one or more places where names [. . .] occur directly'. *Goldfarb 2003* (p. 92f) uses 'placeholder' where Mates says 'counter'. To be sure: the 'counters'—such as circled numerals — are new characters that do not occur in English sentences. Of course, instead of English, which Quine, Mates, and Goldfarb assumed to be interpreted, we use uninterpreted first-order languages and instead of 'counters' we use the single variable x, assumed not to occur in the sentence from which the predicate is 'thought of as having been formed' — reverting to Dummett's words. In the spirit of Dummett, Quine, Mates, Goldfarb, and others, the single variable x may occur multiple times in a predicate. From the fact that the logicians mentioned are all philosophers it should not be inferred that this usage is not found in the writing of mathematicians (*Davis 1958/1982*, pp. 22, 57, 130).

Anyway, a predicate is not a part of a subject–predicate sentence[27] and, in particular, it is not a *predicate letter*, a single uppercase letter with or without subscripts or superscripts as in several recent highly regarded logic books.[28] As implied above, taking the word 'predicate' in the sense of 'predicate letter', there is little to say about existential import: for example, no universalized conditional in the form $\forall x\,(\mathrm{A}x \rightarrow \mathrm{B}x)$ has existential import; the determinant $\exists x\,\mathrm{A}x$ is false under any interpretation with A denoting the null set.

[24] For more on the interrelations of existential import and logical form, see Section 4.

[25] As used here, the adjectives *existential-import* and *import-carrying* are exact synonyms. But *existential-import* works better in attributive position than as a predicate adjective; whereas the second seems to work equally well in either role. The terminology derives from *Goldfarb 2003* (p. 108).

[26] See *Ackrill 1963* (p. 159, passim), *Barnes 2007* (p. 93), and *Corcoran and McGrath 2012*.

[27] In traditional logic and grammar, a predicate is often a part of a subject–predicate 'judgment'. *Hilbert and Ackermann 1928/1938/1950* (p. 44ff) adopted this usage — at least verbally.

[28] *Goldfarb 2003* and *Boolos et al. 2007* are two examples.

A predicate per se is an uninterpreted string in the sense of syntactical string theory.[29] Even under an interpretation, a predicate lacks a truth-value and cannot be used to make a statement. However, under an interpretation INT with universe of discourse U it has an *extension* or *truth-set*: the set of objects in U that satisfy it under INT.[30]

Import-carrying predicates: Consider the existentialized conjunctions $\exists x\,(x = t\ \&\ \mathrm{P}(x))$ used in Gödel's Diagonal Lemma (*Boolos et al. 2007*, p. 221f) — where t is a numeral. These all involve import-carrying predicates. Since t is a numeral, $\exists x\,(x = t)$ is tautological. Thus, $x = t$ is an existential-import predicate and $\exists x\,(x = t\ \&\ \mathrm{P}(x))$ has existential import. If t is the standard numeral denoting the Gödel number of $\mathrm{P}(x)$, then $\exists x\,(x = t\ \&\ \mathrm{P}(x))$ is the 'diagonalization' of $\mathrm{P}(x)$.

As we will see below, the observation that for every $\mathrm{P}(x)$, the diagonalization of $\mathrm{P}(x)$, i.e. $\exists x\,(x = t\ \&\ \mathrm{P}(x))$, has existential import leads to the at-first-surprising result that every sentence containing an individual constant is logically equivalent to a universalized conditional having existential import.

The most obvious examples of import-carrying predicates are tautological predicates, for example, (1) truth-functionally tautological predicates as $(x = t \to x = t)$ and (2) identity-tautological predicates as $x = x$, $(x = t \to t = x)$, etc. — where of course t is any constant term such as a numeral, a 'sum' of two numerals $(n + m)$, and so on. Examples of import-carrying predicates other than tautological trivialities are readily produced. (1) $x = t$, where t is any constant term; (2) $\exists y(x = t(y))$, where $t(y)$ is any term with one free variable y, such as a successor term sy, ss$y$, etc., a 'sum' of two such terms, and so on; (3) $(\exists x\,\mathrm{Q}(x) \to \mathrm{Q}(x))$, where $\mathrm{Q}(x)$ is any predicate whatever, e.g. $\forall y[x \neq (y + (y + 1))]$.

It is obvious that, in any standard first-order language with identity, there are infinitely many import-carrying predicates. But still import-carrying predicates might be isolated special cases that would not be regarded as 'abundant' or 'widespread' in an intuitive sense. Perhaps the traditional doctrine, or 'rule', that no universalized conditional implies its corresponding existentialized conjunction is one that needs tweaking rather than full-fledged refuting.

Much of the rest of this essay is devoted to precise deliberations that will tend to settle such vague questions. One relevant precise point already mentioned is that — although all tautological existentialized conjunctions are obviously implied by their corresponding universalized conditionals — many existentialized conjunctions implied by their corresponding universalized conditionals are not tautological. One example is $\exists x(x = 0\ \&\ x = (x + x))$, which was mentioned in the beginning of this essay.

We ask: exactly how abundant are import-carrying predicates? To answer, let L be any first-order language with any interpretation INT in any universe U. A subset S of U is *definable* [*in L under INT*] iff S is the extension of some predicate $\mathrm{Q}(x)$. S is *import-carrying definable* (respectively, *import-free definable*) iff S is the extension of an import-carrying (respectively, import-free) predicate.[31] A predicate $\mathrm{Q}(x)$ *defines* a subset S of U [in L under INT] iff S is the extension of $\mathrm{Q}(x)$ [in L under INT], i.e. iff S is the set of individuals in U that satisfy $\mathrm{Q}(x)$ [in L under INT].

29 String theory, or pure syntax, was first axiomatized in 1933 in Tarski's truth-definition paper (*Corcoran et al. 1974*).

30 See, for example, *Goldfarb 2003* (p. 94). Moreover, under an interpretation, a predicate may be said to express a *condition* in the same sense that a sentence may be said to express a *proposition* (*Church 1956*, *Corcoran 2009b*), but conditions and propositions play no essential role in the later text, or in other current mathematical discussions of first-order logic.

31 The word 'definable' is used in Tarski's semantic sense of 'definable in an interpretation (model)' as opposed to the syntactic concept 'definable in a theory'. See *Tarski 1983* (pp. xxiii, 118, 194, passim.). The concept of arithmetical definability is a special case of this semantic concept (*Boolos et al. 2007*, pp. 199f, 286f). The concept of definability-in-T found in *Boolos et al. 2007* (p. 207) is a third notion.

Given suitable L and INT, the even-number set is the extension of the import-carrying $\exists y\, x = (y + y)$ *and* of the import-free $\forall y\, x \neq (y + (y + 1))$. This set is typical. Whether the existential-import implication holds is independent of the content (extension) of the antecedent S(*x*) if it is non-empty — just as the existential-import implication's holding is independent of the form and content of the consequent P(*x*), as indicated above.

The *Existential-Import Theorem* is as follows: let L, INT, and U be arbitrary. Every non-empty definable subset of U is *both* import-carrying definable *and* import-free definable.

Whatever 'abundant' means, import-carrying predicates are quite abundant, and no less so than import-free predicates.

The existential-import theorem reduces to two lemmas that are largely unrelated and are best dealt with separately.

The import-carrying-predicate lemma: Let L, INT, and U be arbitrary. Every non-empty definable subset of U is import-carrying definable.

The import-free-predicate lemma: Let L, INT, and U be arbitrary. Every non-empty definable subset of U is import-free definable.

The next section of the paper will prove the Existential-Import Equivalence. The two succeeding sections will prove the import-carrying-predicate lemma and the import-free-predicate lemma. The section following them discusses some equivalence relations useful for thinking about existential import. The final section presents some concluding remarks.

2. The existential-import equivalence

Hence, there can never be surprises in logic. (*Wittgenstein 1922*, 6.1251)

The existential-import equivalence: In any first-order logic, for a universalized conditional to imply the corresponding existentialized conjunction it is necessary and sufficient for the existentialization of the antecedent predicate to be tautological. More succinctly, let S(*x*) and P(*x*) be any predicates in a first-order language:

$$\forall x\,(\mathrm{S}(x) \rightarrow \mathrm{P}(x)) \text{ implies } \exists x\,(\mathrm{S}(x)\,\&\,\mathrm{P}(x)) \text{ iff } \exists x\,\mathrm{S}(x) \text{ is tautological.}$$

There are several proofs independently found by John Corcoran, Sriram Nambiar (personal communication), and Joel Friedman (personal communication). Perhaps the easiest way to see it is as follows.

'If' is almost immediate: obviously the two sentences $\exists x$ S(*x*) and $\forall x$ (S(*x*) $\rightarrow$ P(*x*)) together imply $\exists x$ (S(*x*) & P(*x*)). Moreover, if one of two sentences implying a third is tautological, then the other by itself implies the third. Thus if $\exists x$ S(*x*) is tautological, then $\forall x$ (S(*x*) $\rightarrow$ P(*x*)) implies $\exists x$ (S(*x*) & P(*x*)). QED

'Only if' uses the fact that in order for a given sentence to be tautological it is sufficient for it to be implied by the negation of one of its implicants.

Assume that $\forall x$ (S(*x*) $\rightarrow$ P(*x*)) implies $\exists x$ (S(*x*) & P(*x*)). Since $\exists x$ (S(*x*) & P(*x*)) implies $\exists x$ S(*x*), we have that $\forall x$ (S(*x*) $\rightarrow$ P(*x*)) implies $\exists x$ S(*x*).

Thus to see that $\exists x$ S(*x*) is tautological, it is sufficient to see that the negation $\sim\forall x$ (S(*x*) $\rightarrow$ P(*x*)) implies $\exists x$ S(*x*). But it is obvious that $\sim\forall x$ (S(*x*) $\rightarrow$ P(*x*)) implies $\exists x$ (S(*x*) & $\sim$P(*x*)), which implies $\exists x$ S(*x*). Thus $\exists x$ S(*x*) is tautological. QED

This equivalence is remarkable in several respects. One consequence already mentioned is that whether an existential-import implication holds or fails is entirely independent of the consequent. If a given implication holds and the consequent is replaced by any other predicate, the resulting implication also holds. If it fails and the consequent is replaced by any other predicate, the resulting implication also fails.

Another conclusion that can be drawn, given the obvious fact that there are infinitely many tautological existentials $\exists x\ S(x)$, is that there are infinitely many universalized conditionals that imply their respective corresponding existentialized conjunctions. And given that there are infinitely many predicates $P(x)$, each of those universalized conditionals gives rise — by substitution of the consequent — to infinitely many other universalized conditionals that imply their respective corresponding existentialized conjunctions.

The independence of the consequents, i.e. that substituting arbitrary predicates for the consequent leaves existential import unchanged, suggests that the contrapositive of a universalized conditional implying its corresponding existentialized conjunction need not imply its own corresponding existentialized conjunction — in other words, that the contrapositive of a universalized conditional having existential import need not itself have existential import. This would suggest another source of logically equivalent universalized conditionals some but not all of which imply their respective corresponding existentialized conjunctions. We will see that these suggestions are fruitful.

Corollaries: There are several corollaries of this equivalence that are very close in content. Some people might call them variants of the equivalence. We present a few.

The first existential-import sentence equivalence:

$$\{\forall x\ (S(x) \to P(x)) \to \exists x\ (S(x)\ \&\ P(x))\} \text{ is logically equivalent to } \exists x\ S(x).$$

This is essentially the same as the main result in *Corcoran and Masoud 2014*.
The second existential-import sentence equivalence:

$$\{\forall x\ (S(x) \to P(x)) \to \exists x\ S(x)\} \text{ is logically equivalent to } \exists x\ S(x).$$

The first existential-import tautology:

$$[\{\forall x\ (S(x) \to P(x)) \to \exists x\ (S(x)\ \&\ P(x))\} \to \exists x\ S(x)] \text{ is tautological.}$$

The second existential-import tautology:

$$[\{\forall x\ (S(x) \to P(x)) \to \exists x\ S(x)\} \to \exists x\ S(x)] \text{ is tautological.}$$

This proposition could be called *The Quantified Peirce Law* to recognize its similarity to what *Mates 1972* (p. 107) and others call Peirce's Law in propositional logic.

The next corollary requires a definition: the *Boolean-Existential-Import Schema*, *BEIS*, has as its instances every one-premise argument whose premise is a universalized conditional and whose conclusion is *the* corresponding existentialized conjunction obtained from the universalized conditional by replacing the initial universal quantifier by the existential *and* replacing the conditional connective by the conjunction sign:

$$\frac{\text{Every number } x \text{ is such that (if } A(x),\ \text{then } C(x))}{\text{Some number } x \text{ is such that } (A(x) \text{ and } C(x))}$$

$$\frac{\forall x\ (A(x) \to C(x))}{\exists x\ (A(x)\ \&\ C(x))}$$

The *BEIS Law*: An instance of the argument[32] schema BEIS is valid[33] iff the existentialization $\exists x\, \mathrm{A}(x)$ of the antecedent condition $\mathrm{A}(x)$ is tautological (*Corcoran 2011*).

3. The import-carrying-predicate lemma

> Some logic teachers never reveal that $\exists x\, \mathrm{B}x$ is logically equivalent to its alphabetic variants such as $\exists z\, \mathrm{B}z$. Even worse, they ‘read’ these sentences aloud: ‘Some eckses Bee’ and ‘Some zees Bee’. (*Frango Nabrasa*, personal communication)

The import-carrying-predicate lemma: Let L, INT, and U be arbitrary. Every non-empty definable subset of U is import-carrying definable.

In other words, this lemma says that in any interpretation every non-empty set that is the extension of a predicate is the extension of the determinant predicate of the premise of a true existential-import implication. Excluding the empty set, in any interpretation, the import-carrying definable sets coincide with the definable sets and, thus, import-carrying definable sets are as widespread as they can be.

A transformation: To see this we will first prove that for every predicate $\mathrm{Q}(x)$, the conditional predicate $(\exists x\, \mathrm{Q}(x) \to \mathrm{Q}(x))$ is import-carrying, i.e. that its existentialization $\exists x\, (\exists x\, \mathrm{Q}(x) \to \mathrm{Q}(x))$ is tautological. Because this transformation is so important in this work, it deserves a name: $(\exists x\, \mathrm{Q}(x) \to \mathrm{Q}(x))$ is *the existential qualification of* $\mathrm{Q}(x)$. Of course, for every predicate $\mathrm{Q}(x)$, the conditional predicate $(\exists x\, \mathrm{Q}(x) \to \mathrm{Q}(x))$ is called *an existential qualification.*

Notice that all occurrences of the variable x in the antecedent of an existential qualification are bound but there are free occurrences of x in the consequent and thus in the existential qualification itself. The fact that it is the same variable occurring bound in the antecedent and free in the consequent is irrelevant and thus potentially confusing. In certain contexts therefore it would be convenient to replace the antecedent $\exists x\, \mathrm{Q}(x)$ by one of its logically equivalent alphabetic variants, say $\exists y\, \mathrm{Q}(y)$ as $(\exists y\, \mathrm{Q}(y) \to \mathrm{Q}(x))$, to emphasize logical form. For example, the predicate $[\exists y\, 0 = (y + y) \to 0 = (x + x)]$ is in the same logical form as the predicate $[\exists x\, 0 = (x + x) \to 0 = (x + x)]$, which is an alphabetic variant. However, this proves to complicate the exposition. See Section 4 for more on formal equivalence and logical equivalence in relation to existential import.

Using the new terminology, our first result then will be that every existential qualification is import-carrying.

To see this we first make two useful logical observations.

The first is that every existential implies the existentialization of any conditional whose consequent is the existential’s predicate, i.e. that $\exists x\, \mathrm{Q}(x)$ implies $\exists x\, (\mathrm{P} \to \mathrm{Q}(x))$, no matter whether P is a one-place predicate $\mathrm{R}(x)$ or a sentence. The second observation is that every negation implies the existentialization of any conditional whose antecedent is the sentence negated, i.e. that $\sim\mathrm{P}$ implies $\exists x\, (\mathrm{P} \to \mathrm{R}(x))$, no matter which predicate $\mathrm{R}(x)$ is.

[32] By ‘argument’ we mean ‘sentential premise-conclusion argument’, a two-part system composed of a set of sentences — its premises — and a single sentence – its conclusion. Accordingly, the argument given is unaffected by the order in which the premises are given and some arguments have empty premise sets. Moreover, the conclusion can be among the premises and it is irrelevant whether an argument was ever thought about and a fortiori whether it was ever claimed or even thought to be valid. An argument is valid iff its conclusion is implied by its premise set. Needless to say, this sense of ‘argument’, though widely used in mathematically precise literature (*Boolos et al. 2007*), competes with several other senses (*Mates 1972*, *Corcoran 1989*).

[33] Of course, ‘valid’ is used in the traditional sense that applies only to arguments: an argument is said to be *valid* iff its premise set implies its conclusion –as in *Church 1956*, *Corcoran 1989*, and *Boolos et al. 2007*. Other logicians disapprove of this usage and instead use ‘sound’ — as in *Mates 1972* and *Lemmon 1965/1978* — or they alternate among ‘deductive’, ‘cogent’, ‘correct’, and even ‘correct deductive’ — as in *Goldfarb 2003*.

In other words, any given sentence's negation implies the existentialization of any conditional whose antecedent is the given sentence.

By the first observation, $\exists x\, Q(x)$ implies $\exists x(\exists x\, Q(x) \rightarrow Q(x))$.
By the second observation, $\sim\exists x\, Q(x)$ implies $\exists x(\exists x\, Q(x) \rightarrow Q(x))$.

Thus, $\exists x(\exists x\, Q(x) \rightarrow Q(x))$ is tautological since it is implied by the negation of one of its implicants. This proves that for every predicate $Q(x)$, the existential qualification $(\exists x\, Q(x) \rightarrow Q(x))$ is import-carrying.

To prove the lemma, let L, INT, and U be arbitrary. Let Q be any non-empty definable subset of U. Without loss of generality let Q be defined by $Q(x)$. Since Q is non-empty, $\exists x\, Q(x)$ is true. Thus $(\exists x\, Q(x) \rightarrow Q(x))$ is coextensive with $Q(x)$: a given member of U satisfies the first iff it satisfies the second.[34] Thus $(\exists x\, Q(x) \rightarrow Q(x))$ defines Q.

Thus every non-empty definable subset of U is import-carrying definable. QED

The last part of the proof is model-theoretic semantics, and it is closely related to the fact of first-order logic that $\exists x\, S(x)$ implies $\forall x\, [(\exists x\, S(x) \rightarrow S(x)) \leftrightarrow S(x)]$. There is not much more to the latter than the fact of propositional logic that P implies $[(P \rightarrow Q) \leftrightarrow Q]$.

To see that non-emptiness is required notice that the empty set is not an import-carrying-definable set. The empty set is defined in any interpretation by $x \neq x$ but the existential qualification $(\exists x\, x \neq x \rightarrow x \neq x)$ defines U. In fact, in order for a predicate to be import-free it is necessary and sufficient for it to define the null set under some interpretation.

4. The import-free-predicate lemma

> As Cantor taught us, in a countably infinite universe, there are uncountably many subsets. But, as Tarski taught us, in a countable language interpreted in a countable universe, there are only countably many predicates, thus at most countably many definable subsets; most subsets are undefinable. (Common knowledge)

The import-free-predicate lemma: Let L, INT, and U be arbitrary. Every non-empty definable subset of U is import-free definable.

In this case, the restriction to non-empty subsets is mathematically unnecessary: its inclusion is to emphasize the connection with the import-carrying-predicate lemma. Our task here then is to find, for any given predicate, a coextensive import-free predicate, a coextensive predicate whose existentialization is non-tautological, i.e. false in some interpretation. We give two proofs that analyze the facts in two ways. The first uses a transformation that sheds more light on how abundant or widespread import-free predicates are — without considering what they define.

Another transformation: We will first prove that for every predicate $Q(x)$, the conjunctional predicate $(\exists x \sim Q(x)\ \&\ Q(x))$ is import-free, i.e. that its existentialization $\exists x(\exists x \sim Q(x)\ \&\ Q(x))$ is non-tautological. Because this transformation is so important in this work, it deserves a name: $(\exists x \sim Q(x)\ \&\ Q(x))$ is *the existential-negative conjunctification of* $Q(x)$. Of course, for every predicate $Q(x)$, the conjunctional predicate $(\exists x \sim Q(x)\ \&\ Q(x))$ is called *an existential-negative conjunctification.*

Notice, as with existential qualifications, that all occurrences of the variable x in the first conjunct are bound but there are free occurrences of x in the second conjunct and thus in the existential-negative conjunctification. In certain contexts it would be convenient

[34] The full definition is that one predicate is *coextensive with* another *under* an interpretation with a given universe iff a given member of the universe satisfies the first under the interpretation iff it satisfies the second under that interpretation. This is a three-place relation: X is coextensive to Y under Z. *Boolos et al. 2007* (p. 296ff) use 'coextensive' for a two-place relation as does *Quine 1970/1986* (p. 73ff) — but not for the same relation.

to replace the conjunct $\exists x \sim Q(x)$ by one of its logically equivalent alphabetic variants say $\exists y \sim Q(y)$ as $(\exists y \sim Q(y)\ \&\ Q(x))$ to emphasize logical form. Again, this proves to complicate the exposition.

Using the new terminology, our first result then will be that every existential-negative conjunctification is import-free. We have to show that for every predicate Q(x), there is an interpretation under which $\exists x(\exists x \sim Q(x)\ \&\ Q(x))$ is false. Notice that $\exists x(\exists x \sim Q(x)\ \&\ Q(x))$ is logically equivalent to the conjunction $(\exists x \sim Q(x)\ \&\ \exists x\, Q(x))$. In any INT having a singleton universe, either Q(x) defines the universe — and thus INT falsifies $\exists x \sim Q(x)$ — or $\sim Q(x)$ defines the universe, and thus INT falsifies $\exists x\, Q(x)$.[35]

Thus, we have a one–one transformation that carries each predicate to a closely related predicate that is import-free. This suggests that import-free predicates are widespread. However, this result can be used to get stronger suggestions.

First Proof

Let L, INT, and U be arbitrary. Let Q be any definable subset of U — empty or not. Without loss of generality, let Q be defined by Q(x). Either Q(x) defines the universe or not.

If Q(x) defines the universe U, it is coextensive with $\forall y\, x = y$ or $\exists y\, x \neq y$ according as U is singleton or not. Moreover, both are import-free: their existentializations are informative.

If Q(x) does not define the universe U, $\exists x \sim Q(x)$ is true. Thus $(\exists x \sim Q(x)\ \&\ Q(x))$ is coextensive with Q(x): a given member of U satisfies the first iff it satisfies the second. Thus $(\exists x \sim Q(x)\ \&\ Q(x))$ defines Q and is import-free, as we saw above.

Thus, whether Q(x) defines the universe or not, it is coextensive with an import-free predicate. Thus, every non-empty definable subset of U is import-free definable.[36] QED

Second Proof

Let L, INT, and U be arbitrary. Let Q be any definable subset of U — empty or not. Either U is a singleton or not.

Let U be a singleton. Then Q is U or {}. If Q is U, then Q is defined by $\forall y\, x = y$, which is import-free, as we saw above. If Q is {}, then Q is defined by $\exists y\, x \neq y$, which is import-free, as we saw above.

Now we treat non-singleton universes. Consider $\exists y\, x \neq y$, which defines the universe in non-singleton interpretations and whose existentialization is false in singleton universes. This predicate is import-free and so is any other predicate that implies it.[37] Without loss of generality, let Q be defined by Q(x). $(\exists y\, x \neq y\ \&\ Q(x))$ implies $\exists y\, x \neq y$ and it is thus import-free. Moreover, for every R(x), in any non-singleton universe $(\exists y\, x \neq y\ \&\ R(x))$ is coextensive with R(x)). Hence, $(\exists y\, x \neq y\ \&\ Q(x))$ is coextensive with Q(x). The lemma is proved for non-singleton universes.

Thus, for every first-order language L, any interpretation INT in a universe U, every non-empty definable subset of U is import-free definable. QED

5. Equivalence relations

> In every proposition and in every inference there is, besides the particular subject-matter concerned, a certain form, a way in which the constituents of the proposition or inference are put together. (*Bertrand Russell 1914*, p. 47)

[35] Of course, by INT *falsifies* S we mean that S is false under INT (*Goldfarb 2003*, p. 283).

[36] We proved the stronger result that every definable subset of U is import-free definable, but the weaker result stated is sufficient for purposes of the above-stated Existential-Importance Theorem. The same applies to the second proof.

[37] For any predicate pair, a *corresponding sentence pair* is obtained by replacing all free occurrences of the variable by a constant occurring in neither. Saying that the first predicate of a pair implies the second is saying that the first sentence of the corresponding sentence pair implies the second.

(1) *Formal equivalence*: Two sentences are defined to be *formally equivalent* or *equivalent in form* iff they have the same logical form (*Corcoran 2004*, p. 445).[38] For example, the following are formally equivalent to each other:

$$\forall x(x = 0 \rightarrow x = (x + x))$$

$$\forall x(x = 1 \rightarrow x = (x + x))$$

$$\forall x(x = 1 \rightarrow x = (x \times x))$$

$$\forall y(y = 0 \rightarrow y = (y + y))$$

The second has the individual constant '1' where the first has the individual constant '0'. The third has the two-place function symbol '×' where the second has the two-place function symbol '+'. The last is an alphabetic variant of the first.

Thus, two sentences are formally equivalent if some 1–1 category-preserving function from the set of non-logical constants of the language onto itself carries one sentence exactly into the other. Any two formally equivalent sentences have exactly the same number of character occurrences, for each logical constant exactly the same number of occurrences of that constant, the same number of variable occurrences, and so on.[39]

A property of sentences is said to be *formal* if it belongs to every sentence formally equivalent to one it belongs to, i.e. if formal equivalence preserves it. Being a universal, being an existential, being a conjunction, a negation, a conditional, a universalized conditional, an existentialized conjunction, and the like are all formal properties as are being tautological and being informative. As mentioned above in different words, being a universalized conditional with existential import is a formal property. Formal equivalence preserves existential import in the sense that every formal equivalent of a universalized conditional with existential import also has this property, i.e. is a universalized conditional with existential import.

This treatment of formal equivalence, like the similar Tarski–Givant treatment, is based on the *relation* misleadingly expressed 'has the same logical form as', but it does not postulate or presuppose 'forms' in any of various senses.[40]

(2) *Logical equivalence*: Logical equivalence does not preserve existential import: not every logical equivalent of a universalized conditional with existential import also has this property. This was exemplified above with contrapositives. This remark answers the question: does every universalized conditional logically equivalent to some universalized conditional with existential import have existential import?

However, what was not discussed was how extreme the lack of preservation is. For example, is every universalized conditional whose antecedent predicate is import-carrying logically equivalent to some universalized conditional whose antecedent predicate is import-free? In other words, is every universalized conditional with existential import logically equivalent to some universalized conditional without existential import? As far as we know, this is an open question.

If this is answered no, we can ask instead which universalized conditionals with existential import are logically equivalent to universalized conditionals without existential import.

[38] In other words, using the poorly chosen expression 'almost identical' in the sense of *Tarski and Givant 1987* (p. 43), two sentences are *formally equivalent* or *equivalent in form* iff they are either almost identical or one is an alphabetic variant of a sentence almost identical to the other.

[39] For further information see *Russell 1919* (p. 199) and *Corcoran 1989* (p. 27f).

[40] Several senses of 'form' play important roles in *Russell 1919* (p. 199), *Church 1956* (pp. 2, 10ff), *Goldfarb 2003* (pp. 5, 48, 150, passim), and *Dutilh Novaes 2011* (passim).

Of course, once the above questions are raised, one immediately asks the converse of the first: is every universalized conditional whose antecedent predicate is import-free logically equivalent to some universalized conditional whose antecedent predicate is import-carrying? In other words, is every universalized conditional without existential import logically equivalent to a universalized conditional with existential import? Perhaps surprisingly, this is not an open question. The answer is yes. However, some proofs also show that the question is less interesting than might have been thought. There is less to this than meets the eye.

Import-creation corollary: Every universalized conditional whose antecedent predicate is import-free is logically equivalent to a universalized conditional whose antecedent predicate is import-carrying. In other words, every universalized conditional without existential import is logically equivalent to a universalized conditional with existential import.
Proof: Assume that $\forall x\,(\mathrm{S}(x) \to \mathrm{P}(x))$ does not imply $\exists x\,(\mathrm{S}(x)\,\&\,\mathrm{P}(x))$. Now consider the universalized conditional whose antecedent predicate is $x = x$ and whose consequent predicate is $(\mathrm{S}(x) \to \mathrm{P}(x))$, i.e. $\forall x\,\{x = x \to (\mathrm{S}(x) \to \mathrm{P}(x))\}$. The required logical equivalence is obvious; it is a special case of the fact that for every $\mathrm{P}(x)$, $\forall x\,\mathrm{P}(x)$ is logically equivalent to $\forall x\,\{x = x \to \mathrm{P}(x)\}$. QED

The implied existential is $\exists x\,\{x = x\,\&\,(\mathrm{S}(x) \to \mathrm{P}(x))\}$.

Notice that the Import-Creation Corollary implies that absolutely every universalized conditional — regardless of antecedent predicate — is logically equivalent to a universalized conditional whose antecedent predicate is import-carrying. In other words, every universalized conditional is logically equivalent to a universalized conditional with existential import.

This brings us to the result announced above about sentences containing an individual constant.

The individual-constant import-creation lemma: Every sentence containing an individual constant is logically equivalent to a universalized conditional having existential import.
Proof: Let P be any sentence containing an individual constant, say t. Without loss of generality, assume that x does not occur in P and that $\mathrm{P}(x)$ is the predicate obtained by replacing every occurrence of t by x. As noted by *Boolos et al. 2007* (p. 221), it is easy to see that P is logically equivalent to $\exists x\,(x = t\ \&\ \mathrm{P}(x))$. But it is also easy to see that the latter existentialized conjunction, $\exists x\,(x = t\ \&\ \mathrm{P}(x))$, is in turn logically equivalent to its own corresponding universalized conditional $\forall x\,(x = t \to \mathrm{P}(x))$, which has existential import. QED

This result admits of generalization in view of the fact that every sentence is logically equivalent to one containing an individual constant: P is logically equivalent to $(\mathrm{P}\,\&\,t = t)$. Thus, we have proved:

The strong import-creation corollary: Every sentence is logically equivalent to a universalized conditional whose antecedent predicate is import-carrying. In other words, every sentence — whether a universalized conditional or not — is logically equivalent to a universalized conditional with existential import.

(3) *Extensional equivalence*: We move on to another equivalence relation useful in surveying the distribution of existential import and in seeing how widespread it is. Two universal sentences are defined to be *extensionally equivalent [with each other] under a given interpretation* INT iff their predicates define the same set under INT. If two universal sentences are both true, then their predicates define the universe and thus they are extensionally equivalent. In addition, if they have different truth-values, they are not extensionally equivalent. However, if they are both false, it is necessary to dig a little deeper.

Two false universal sentences are extensionally equivalent iff they have the same counterexamples: for any two predicates S(x) and P(x), S(x) and P(x) define the same subset of the universe iff their negations ∼S(x) and ∼P(x) define the same subset.

Our examples use standard first-order logic with the class of numbers [non-negative integers] as universe of discourse. A number n is a *counterexample* for a universal sentence $\forall x\,$P(x) iff n satisfies ∼P(x). In some familiar cases, logically equivalent false sentences have the same counterexamples. 'Every number that is not even is prime' and 'Every number that is not prime is even' are both *counterexemplified* by the non-prime odd numbers.

Moving along a spectrum, we find cases that share some but not all counterexamples. 'Every number divides every other number' is counterexemplified by every number except one, whereas its equivalent 'Every number is divided by every other number' is counterexemplified by every number except zero. On the other end of the spectrum there are cases having no counterexamples in common: 'Every even number precedes every odd number' is counterexemplified only by even numbers, whereas its equivalent 'Every odd number is preceded by every even number' is counterexemplified only by odd numbers. One easy result is that, given any non-empty finite set of numbers, every false universal sentence is logically equivalent to another counterexemplified exclusively by numbers in the given set.

If n is a numeral, then P(n) and Q(n) are *corresponding instances* of universal sentences $\forall x\,$P(x) and $\forall x\,$Q(x). As seen above, it is not necessary for corresponding instances of logically equivalent universal sentences to be logically equivalent. The above phenomenon is quite common; indeed every non-tautological universal sentence is in the same logical form as a false universal sentence that is logically equivalent to another false universal having a different set of counterexamples.[41]

Let us return to the main issue: how extensional equivalence can shed light on how widespread existential import is. Due to space limitations, we will give only two results both easily proved as corollaries of the above considerations. Actually, both are essentially restatements of previous results.

The extensional import-creation corollary: Under any given interpretation, every universal sentence is extensionally equivalent under that interpretation to some universalized conditional having existential import.

The extensional import-destruction corollary: Under any given interpretation, every universal sentence whose predicate does not define the null set is extensionally equivalent under that interpretation to some universalized conditional not having existential import.

(4) *Biextensional equivalence*: We move on to yet another equivalence relation useful in surveying the abundance and distribution of existential import. Two universalized conditional sentences are defined to be *biextensionally equivalent [with each other] under a given interpretation* INT iff their antecedent predicates define the same set under INT and their consequent predicates define the same set under INT.

The import-carrying-predicate lemma — that under any INT, every non-empty definable subset of U is import-carrying definable — implies that under any INT, every universalized conditional is biextensionally equivalent to some universalized conditional having existential import.

In other words, this lemma implies the following.

The biextensional import-creation corollary: In any interpretation, every universalized conditional without existential import is biextensionally equivalent to some universalized conditional with existential import.

[41] This and related results are proved in *Corcoran 2005*.

6. Enthymemic implications

> Two circles intersect if each has its center on the other's circumference. (One of Euclid's enthymemic postulates: *Elements* I.1)

Let L be any suitable first-order language with any interpretation INT in any universe U. Let A and C be distinct one-place predicate letters and let P, Q, and R be sentences. Further, let us say that one given sentence *materially implies* [*under* INT] a second given sentence iff the first is false or the second true, i.e. iff it is not the case that the first is true and the second false. Thus P materially implies Q iff the conditional (P → Q) is true. Notice that 'true' and 'false' are elliptical for 'true under INT' and 'false under INT'.

Taken literally, the following is misleading if not incoherent: 'P materially implies Q if we assume R'. The fact of the matter is that whether P materially implies Q is entirely determined by the truth-values of P and Q under INT: it has nothing to do with what assumptions we may or may not make. The quoted sentence might be generously interpreted as saying: '(P & R) materially implies Q'. Often the context does not warrant such generosity.

Moreover, a statement containing an expression like 'we assume R' implicates or presupposes that R is informative since it is pointless to assume a tautology.

Similar remarks apply to 'P materially implies Q only if we assume R'. Here one generous interpretation would be 'P does not materially imply Q but (P & R) does materially imply Q'.

Now consider the universalized conditional sentence ∀*x* (A*x* → B*x*) and the corresponding existentialized conjunction ∃*x* (A*x* & B*x*). As noted above, it is obvious that ∀*x* (A*x* → B*x*) does not imply ∃*x* (A*x* & B*x*). One countermodel takes A to denote the null set and B the universe U. Notice that if INT takes A to denote a non-empty set, then ∀*x* (A*x* → B*x*) *materially* implies ∃*x* (A*x* & B*x*) — under INT. However, as noted, ∀*x* (A*x* → B*x*) does not [sc. *logically*] imply ∃*x* (A*x* & B*x*).

It is also obvious that the conjunction (∀*x* [A*x* → B*x*] & ∃*x* A*x*) does imply ∃*x* [A*x* & B*x*]. Recognizing discussion in traditional logic texts, we could express the combination of these two obvious facts about ∀*x* [A*x* → B*x*] and ∃*x* [A*x* & B*x*] by saying that the first *enthymemically implies* the second.

To be precise, we stipulate that given any two sentences, the first *enthymemically implies* the second iff the first does not imply the second but there is a third whose conjunction with the first does imply the second. Thus, for any sentences X, Y: X enthymemically implies Y iff X does not imply Y but there is a third sentence Z, such that (Z & X) implies Y. Notice that this definition is deliberately *not* intended to explicate any traditional concept; it is a stipulation for explanatory and heuristic purposes. For example, it follows almost immediately that for any sentences X, Y: if X does not imply Y, then X enthymemically implies Y. For the third sentence Z, we can take (X → Y).

To complete the stipulations, for any sentences X, Y, Z: Z is an *enthymemic premise for* X *relative to* Y iff X does not imply Y but (Z & X) implies Y. It is obvious that for any sentences X, Y, Z: if Z is an *enthymemic premise for* X *relative to* Y then Z is informative (non-tautological). It is also clear that for any two predicates S(*x*) and P(*x*), if the universalized conditional ∀*x* (S(*x*) → P(*x*)) enthymemically implies the corresponding existentialized conjunction ∃*x* (S(*x*) & P(*x*)), then ∃*x* S(*x*) is informative (non-tautological).

Enthymemic-Implication Theorem: For any two predicates S(*x*) and P(*x*), if the universalized conditional ∀*x* (S(*x*) → P(*x*)) enthymemically implies the corresponding existentialized conjunction ∃*x* (S(*x*) & P(*x*)), then (1) every enthymemic premise for ∀*x* (S(*x*) → P(*x*)) relative to ∃*x* (S(*x*) & P(*x*)) implies ∃*x* S(*x*) and (2) ∃*x* S(*x*) is informative.

Before giving our proof, it is convenient to notice an equivalence not often noted in the literature. Let S(x) and P(x) be any predicates: $\{\forall x\,[S(x) \to P(x)] \to \exists x\,S(x)\}$ is logically equivalent to $\exists x$ S(x). This follows from the second existential-import tautology above, namely: $[\{\forall x\,[S(x) \to P(x)] \to \exists x\,S(x)\} \to \exists x\,S(x)]$ is tautological.

Proof: Let S(x) and P(x) be any two predicates. Assume that the universalized conditional $\forall x\,[S(x) \to P(x)]$ enthymemically implies the corresponding existentialized conjunction $\exists x\,[S(x)\,\&\,P(x)]$. By the Existential-Import Equivalence, $\exists x$ S(x) is informative (non-tautological).

Let Q be an enthymemic premise for $\forall x\,[S(x) \to P(x)]$ relative to $\exists x\,[S(x)\,\&\,P(x)]$. We must show that Q implies $\exists x$ S(x).

By hypothesis we have that $\forall x\,[S(x) \to P(x)]$ does not imply $\exists x\,[S(x)\,\&\,P(x)]$ but $\{\forall x\,[S(x) \to P(x)]\ \&\ Q\}$ implies $\exists x\,[S(x)\,\&\,P(x)]$. From the assumption that $\forall x\,[S(x) \to P(x)]\ \&\ Q\}$ implies $\exists x\,[S(x)\,\&\,P(x)]$ it follows that Q implies $\{\forall x\,[S(x) \to P(x)] \to \exists x\,[S(x)\,\&\,P(x)]\}$. Thus Q implies $\{\forall x\,[S(x) \to P(x)] \to \exists x\,S(x)\}$.

However $\{\forall x\,[S(x) \to P(x)] \to \exists x\,S(x)\}$ implies $\exists x$ S(x). QED

The Enthymemic-Implication Theorem might seem to agree with the textbooks but it does not. See, for example, *Reichenbach 1947*. In the first place, many textbooks give the impression that, in every case, the universalized conditional $\forall x\,[S(x) \to P(x)]$ enthymemically implies the corresponding existentialized conjunction $\exists x\,[S(x)\,\&\,P(x)]$. However, we saw that this is not true: the only universalized conditionals $\forall x\,[S(x) \to P(x)]$ that enthymemically imply the corresponding existentialized conjunctions $\exists x\,[S(x)\,\&\,P(x)]$ are those with import-free antecedent predicates. In the second place, many textbooks give the impression that in every case where the universalized conditional $\forall x\,[S(x) \to P(x)]$ enthymemically implies the corresponding existentialized conjunction $\exists x\,[S(x)\,\&\,P(x)]$, the enthymemic premise is $\exists x$ S(x). However, we saw above that any of the infinitely many implicants of $\exists x$ S(x) will serve just as well.

In the third place, many textbooks give the impression that in *no* case does the universalized conditional $\forall x\,[S(x) \to P(x)]$ imply the corresponding existentialized conjunction $\exists x\,[S(x)\,\&\,P(x)]$. For example, *Cohen and Nagel 1993* (p. 41) wrote: 'Universals do not affirm the existence of any individuals, but simply deny the existence of certain kinds of individuals'. It is clear from the immediate context that by 'affirm' they mean 'imply' and that by 'universal' they mean 'universalized conditional'. They make the same mistaken point using other words on later pages. For example, on the next page they refer to their 'conclusion that universals do not imply the existence of verifying instances'.[42] They use almost the same words in the final paragraph of the next page (*Cohen and Nagel 1993*, p. 43).

In a sentence carrying over to the next page they say: 'we cannot infer the truth of a particular proposition from universal premises alone; unless indeed we tacitly take for granted the existence [sc. non-emptiness] of the classes denoted by the terms of the universal proposition'.[43] This passage repeats the mistake of saying that no existential is implied by the corresponding universal. Moreover, it adds a new mistake: thinking that what we 'take for granted' has something to do with whether an implication holds. If an implication does not hold, there is nothing anyone can 'take for granted' that can change the situation.

[42] This is another sense of 'instance': from the context we infer that 'a verifying instance of a universal [sc. interpreted universalized conditional]' is an object in the universe of discourse that verifies the antecedent predicate — roughly speaking.

[43] Notice they say: 'we cannot infer *the truth of* a particular proposition from universal premises alone [. . .]' — italics added. But it is beyond the scope of this paper to discuss the important differences between inferring a proposition and inferring the truth of a proposition or discussing the extent to which these distinctions are respected by Cohen and Nagel.

The quoted sentence might be generously interpreted as saying that particular propositions are enthymemically implied by the corresponding universal premises and that in every case the only enthymemic premise is the existentialization of the universal's antecedent predicate. However, that construal would have Cohen and Nagel making two mistakes already described.

The excellent — and in other ways technically impeccable – *Goldfarb 2003* is another book that could give some readers the impression that it says that no universalized conditional has existential import. On page 108, it reads as follows.

> Whether [. . .] ordinary language 'all'-statements carry *existential import* — require for their truth the existence of a value for '*x*' that makes the antecedent of the conditional true — is beside the point for us. [. . .] Below we shall always use such 'all'-statements without existential import; we shall interpret them as simple universal quantifications [sc. universalized conditionals].

Apparently, this implicitly suggests that no universalized conditional has existential import. If 'all'-statements are treated as universalized conditionals and they are used without existential import, then universalized conditionals cannot have existential import. The suggestion is all the more likely given the absence of comments to the contrary. The fact that Goldfarb had just used the predicate '*x* is self-identical' a few pages earlier on page 94 shows he could easily have cited universalized conditionals such as 'Everything *x* is such that: if *x* is self-identical, *x* is material', or even 'Everything *x* is such that: if *x* is self-identical, *x* is self-identical', which 'require for their truth the existence of a value for "*x*" that makes the antecedent of the conditional true'. In *Goldfarb 2003*, there is no sign of awareness of import-carrying predicates — by that or any other description. Nevertheless, Goldfarb did not intend to deny the existence of universalized conditionals with existential import in the quoted passage or elsewhere (personal communication).

It is probably worth noting that Cohen and Nagel never mention the fact, emphasized by *Reichenbach 1947* (p. 92f) and *Suppes 1957/1999* (p. 68), that the universal $\forall x\ P(x)$ implies the corresponding existential $\exists x\ P(x)$ in every case.[44] On the other hand, *Cohen and Nagel 1993* (p. 41f) use the expression 'existential import' repeatedly but Reichenbach and Suppes studiously avoid it.

None of the three books notice the fact that universals such as $\forall x\ Ax$, which implies $\exists x\ Ax$, are not in the same logical form as universalized conditionals such as $\forall x\ [Ax \rightarrow Bx]$, which does not imply $\exists x\ [Ax \,\&\, Bx]$. Nor does either notice the fact that existentials such as $\exists x\ Ax$, which $\forall x\ Ax$ implies, are not in the same logical form as existentialized conjunctions such as $\exists x\ [Ax \,\&\, Bx]$, which $\forall x\ [Ax \rightarrow Bx]$ does not imply. *Suppes 1957/1999* does not even mention the fact that $\forall x\ [Ax \rightarrow Bx]$ does not imply $\exists x\ [Ax \,\&\, Bx]$.

Like Cohen and Nagel, *Goldfarb 2003* does not see fit even to mention the fact that every universal implies the corresponding existential.[45]

Another point that is important for teaching existential import and that is often omitted in textbooks is the disanalogy between grammatical form and logical form. In particular although the existential proposition corresponding to the universal proposition expressed by the English sentence 'Every number precedes some odd number' is expressed by the sentence obtained by replacing 'every' by 'some', namely, 'Some number precedes some

[44] Compare this with *Church 1956* (p. 187, Theorem *331).

[45] However, *Goldfarb 2003* (p. 180) comes close: 'Thus, a universally quantified schema implies each of its instances, and an existentially quantified schema is implied by each of its instances'. Note that Goldfarb is not using 'schema', 'instance', and 'implies' in any of the usual senses catalogued in (*Corcoran 2006*). For Goldfarb not every schema has instances, some schema instances have instances, some schemata imply their instances, and some schemata are implied by their instances.

odd number', a simple replacement of 'every' by 'some' does not have the same effect on sentences expressing universalized conditionals.

The sentence 'Some even number precedes some odd number' does not express the corresponding existential of the universal proposition expressed by 'Every even number precedes some odd number'. That proposition would be expressed by 'Some number, if even, precedes some odd number'.

Likewise, the sentence 'Every even number precedes some odd number' does not express the corresponding universal of the existential proposition expressed by 'Some even number precedes some odd number'. That proposition would be expressed by 'Every number is even and precedes some odd number'.[46]

Textbook treatment of existential import suffers many other flaws whose corrections are beyond the scope of this essay. The textbook 'rule' that no existentialized conjunction is implied by the corresponding universalized conditional has many exceptions, many of which are trivial but many of which are far from trivial — as we have seen.

7. Concluding remarks

> If you by your rules would measure what with your rules doth not agree, forgetting all your learning, seek ye first what its rules may be. (Richard Wagner, *Die Meistersinger*)

Exemplifying the widely heralded failure of existential import in modern logic, we can point out that

> 'Every number that is even precedes some odd number'

does not imply

> 'Some number that is even precedes some odd number'.

In addition, analogously we note that

$$\forall x[\mathrm{E}x \rightarrow \exists y(x < y \,\&\, \mathrm{O}y)] \text{ does not imply } \exists x[\mathrm{E}x \,\&\, \exists y(x < y \,\&\, \mathrm{O}y)].$$

However, replacing Ex, 'x is even', by the coextensive predicate $(\exists y\, x = (y + y))$, '$x$ is the sum of a number with itself', an implication results.

> 'Every number that is the sum of a number with itself precedes some odd number'

does imply

> 'Some number that is the sum of a number with itself precedes some odd number'.

$$\forall x(\exists y\, x = (y + y) \rightarrow \exists y(x < y \,\&\, Oy)) \text{ implies } \exists x(\exists y\, x = (y + y) \,\&\, \exists y(x < y \,\&\, Oy)).$$

This is typical. As shown in this paper, whenever a universalized conditional proposition fails to imply the corresponding existentialized conjunction, there is another universalized conditional — with the same consequent and a coextensive antecedent — that does imply its corresponding existentialized conjunction. The exception, of course, is the case where the antecedent of the universalized conditional defines the null set.

[46] *Suppes 1957/1999* (p. 49) and *Goldfarb 2003* (p. 105) make closely related points well worth study.

Although this essay concerns exclusively one-place predicates that define sets of individuals under interpretations, similar results hold for two-place predicates.

$$\forall x \forall y[(x = 0 \,\&\, y = 1) \to x < y] \text{ implies } \exists x \exists y\, [(x = 0 \,\&\, y = 1) \,\&\, x < y].$$

$$\forall x \forall y\, [\mathrm{x} < y \to (x = 0 \,\&\, y = 1)] \text{ does not imply } \exists x \exists y\, [x < y \,\&\, (x = 0 \,\&\, y = 1)].$$

As might have been anticipated, we can prove the following *Two-place Existential-Import Equivalence.*

$$\forall xy\, [\mathrm{S}(x, y) \to \mathrm{P}(x, y)] \text{ implies } \exists xy\, [\mathrm{S}(x, y) \,\&\, \mathrm{P}(x, y)] \text{ iff } \exists xy\, \mathrm{S}(x, y) \text{ is tautological.}$$

From here, it is easy to formulate and prove the general case for *n*-place predicates.

Acknowledgements

In memory of Professor Robert F. McNaughton and his teacher Professor Willard V. O. Quine, inspiring teachers who taught far more than what they were taught. We are especially grateful to colleagues around the world including: Edgar Andrade, Universidad del Rosario, Bogotá; Gabriela Fulugonia, University of Buenos Aires; Leonard Jacuzzo, Fredonia University; Calvin Jongsma, Dordt College; Justin Legault, Ottawa, Canada; Edwin Mares, Victoria University, Wellington; Sriram Nambiar, Indian Institute of Technology, Madras; José Miguel Sagüillo, University of Santiago de Compostela; and Michael Scanlan, Oregon State University. Valuable help and advice was also received from Mark Brown, Newton da Costa, William Demopoulos, William Frank, Joel Friedman, James Gasser, Marc Gasser, Warren Goldfarb, Luis Estrada González, Anil Gupta, David Hitchcock, Wilfrid Hodges, Joaquin Miller, Bo Mou, Frango Nabrasa, Jacek Pasniczek, Wagner Sanz, Wilfried Sieg, Kevin Tracy, Alasdair Urquhart, Albert Visser, Stanley Ziewacz, Jan von Plato, George Weaver, Justin Zylstra, and others.

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