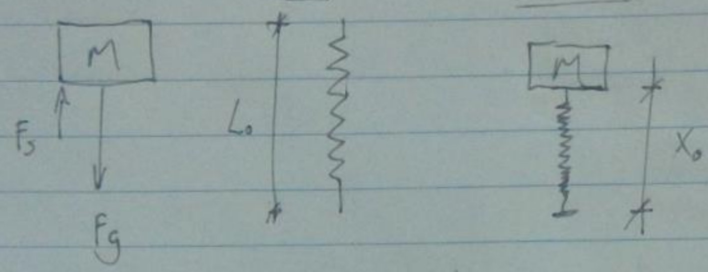
Assignment

1.)

FBD:

Static: Dynamic:

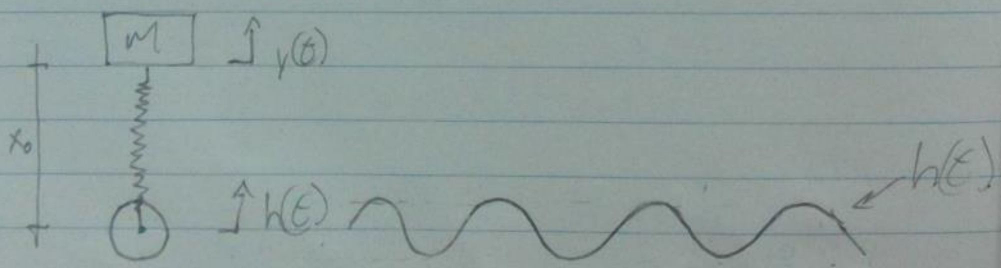


Now... $\Sigma F = m\cancel{a}^{0} = 0$ @ Equil.

$\therefore F_s = F_g = mg$

$F_s = k[L_0 - X_0] \quad \therefore k[L_0 - X_0] = mg$

Now...



Now... $F_s = k[L_0 - L] = k[L_0 - (X_0 + y - h)]$ where L = current length

$\therefore \Sigma F = -mg$

recall... $mg = k(L_0 - X_0)$ →

$\therefore \Sigma F = -mg + k[L_0 - (X_0 + y + h)]$

$= \cancel{-mg} + \cancel{k[L_0 - X_0]} - k[y - h]$

$\therefore \Sigma F = -k[y - h] = ma$

$-ky + kh = m\ddot{y}$

$$\therefore \frac{d^2 y(t)}{dt^2} + \frac{k}{m}[y(t)] = \frac{k}{m}[h(t)]$$

2) Homogeneous soln:

$$y''(t) + \frac{k}{m} y(t) = 0$$

$$\therefore \lambda^2 + \frac{k}{m} = 0$$
$$\lambda^2 = -\frac{k}{m}$$
$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

3) $y''(t) + \frac{k}{m}y(t) = \frac{k}{m}e^{i\omega t}$

let $h(t) = e^{i\omega t}$

Let $y(t) = Ae^{i\omega t}$
$\therefore y'(t) = A\omega e^{i\omega t}$
$y''(t) = A(\omega i)^2 e^{i\omega t}$

$$\therefore (i\omega)^2 Ae^{i\omega t} + \frac{k}{m}Ae^{i\omega t} = \frac{k}{m}e^{i\omega t}$$

$\frac{k}{m} = \omega_0^2$

$\omega_0 \equiv$ Natural frequency

$$\therefore -\omega^2 A + \frac{k}{m}A = \frac{k}{m}$$

$$A\left(\frac{k}{m} - \omega^2\right) = \frac{k}{m}$$
$$A\left(\frac{\omega^2 k - \omega^4 m}{m\omega^2}\right) = \frac{k}{m}$$

$\therefore -\frac{k}{m}y'' + \frac{k}{m}y = \frac{k}{m}x$

$$A = \frac{k}{m}\left(\frac{m\omega^2}{\omega^2 k - \omega^4 m}\right)$$

$$= \frac{k\omega^2}{\omega^2 k - \omega^4 m}$$

$$A(\omega) = \frac{1}{1 - \frac{m}{k}\omega^2}$$

$\therefore$ At $\omega^2 = \frac{k}{m}$, (i.e. $\omega = \sqrt{\frac{k}{m}} \equiv \omega_0$, the natural frequency of the system.)

$A(\omega) \to \infty$. This is due to resonance as the exciting frequency equals the natural frequency of the spring-mass system.

Part 1B

Friday, 28 October 2016 6:10 PM

$M = 200kg$

$k = 1800\ N/m$

$L_0 = 0.25\ m$

4.) See Appendix A

5.) See Appendix A

6.) The system is in sustained oscillation since there is no dampening occuring to reduce the steady state response. Damping is considered a frictional force, & thus dissapates energy. No damping is present in this case & as the exciting frequency, ω, tends towards 3 rads/sec, $H(\omega) \to \infty$, creating resonance, since the natural frequency, $\omega_0 = 3$.
No. Though @ higher frequencies it might by sufficient.

Part 2

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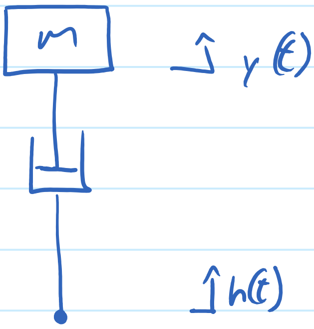


7) $F_d = c\left[\frac{dy}{dt} - \frac{dh}{dt}\right]$

Now incorporating Part 1 results:

$$\Sigma F = -k(y-h) - c\left(\frac{dy}{dt} - \frac{dh}{dt}\right) = m\frac{d^2y}{dt^2}$$

$$\therefore \quad m\left(\frac{d^2y}{dt^2}\right) + c\left(\frac{dy}{dt}\right) + ky = kh + c\left(\frac{dh}{dt}\right)$$

std form

$$\therefore \quad \underbrace{\frac{d^2y}{dt^2} + \frac{c}{m}\left[\frac{dy}{dt}\right] + \frac{k}{m}\left[y(t)\right]}_{\text{Homogeneous}} = \underbrace{k\left[h(t)\right] + c\left[\frac{dh}{dt}\right]}_{\text{Forcing terms}}$$

8) See Appendix B

9) assuming $y(t) = Ae^{\lambda t}$

$$\frac{dy}{dt} = A\lambda e^{\lambda t}$$

$$\frac{d^2y}{dt^2} = A\lambda^2 e^{\lambda t}$$

Also look for $A \neq 0$ $\quad e^{\lambda t} \neq 0$

$$\therefore \quad mA\lambda^2 e^{\lambda t} + cA\lambda e^{\lambda t} + kAe^{\lambda t} = 0$$

$$\therefore \quad m\lambda^2 + c\lambda + k = 0$$

$$\therefore \quad \lambda = \frac{c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$\therefore$ IF:

$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} > 0 \Rightarrow$ real roots

" " $< 0 \Rightarrow$ imaginary roots

" " $= 0 \Rightarrow$ repeated roots

@ $c=0$ (As before)

$\lambda = \pm i\sqrt{\frac{k}{m}}$

While $c \neq 0$ let $\alpha = \frac{c}{2m}$

$\therefore \lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm (\alpha^2 - \omega_0)i$ where $\omega_0 = \sqrt{\frac{k}{m}}$

Now when $\alpha < \omega_0$

complex conj.

$y(t) = Ae^{-\alpha \pm \sqrt{\alpha^2 - \omega_0^2}}$

$y(t) = A_1 e^{(\alpha + \sqrt{\alpha^2 - \omega_0^2})t} + A_2 e^{(-\alpha - \sqrt{\alpha^2 - \omega_0^2})t}$

$= e^{-\alpha t}\left(C_1 \sin\left[(\omega_0^2 - \alpha^2)t\right] + C_2 \cos\left[(\omega_0^2 - \alpha^2)t\right]\right)$

Where $\alpha > \omega_0$

$\lambda_1 = \alpha + \sqrt{\alpha^2 - \omega_0^2}$

$\lambda_2 = \alpha - \sqrt{\alpha^2 - \omega_0^2}$ still $y(t) = Ae^{\lambda t}$

$\therefore y(t) = C_1 e^{(-\alpha + \sqrt{\alpha^2 - \omega_0^2})t} + C_2 e^{(-\alpha - \sqrt{\alpha^2 - \omega_0^2})t}$

Where $\alpha = \omega_0$

$\lambda = -\alpha$ ← repeated roots

$\therefore y(t) = C_1 e^{-\alpha t} + tC_2 e^{-\alpha t}$

For $C = 200$

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{\left(\frac{200}{400}\right)^2 - \frac{1800}{200}}$$
$$= \sqrt{\frac{1}{4} - 900}$$
$$= i\sqrt{\frac{3599}{4}}$$

$\therefore$ underdamped

$\therefore$ complx roots.

$C = 400$

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{1 - 900}$$
$$= i\sqrt{899}$$

$\therefore$ underdamped complx roots

$C = 1800$

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{8100 - 900}$$
$$= \sqrt{7200}$$

$\therefore$ overdamped. real roots

In the overdamped case, the damping coefficient, c, is so large that it inhibits the systems oscillitory response. Both exponents become negative forcing every solution to asymptote to $y = 0$. Much like the iconic "Ikea drawer system". The higher the value for c in this case, the longer the system takes to settle @ equilibrium.

As explained in question 6, an underdamped syst will continue oscillation, though will come to rest @ equilibrium eventually if some damping is present. In the case of a relatively small "c", a decreasing amplitude would be observed in the system response as the energy is converted to heat. The threshold for oscillitory behaviour is called the critical damping coëficient, C_c. This occurs where:

$$C^2 = 4mk.$$

In critical damping, as in the overdamping case, 'c' is high enough to induce enough friction to prohibit oscillation of the harmonic system. When contrasting the overdamped & critical cases for the same mass & spring constants one would notice that the critical case has the fastest decay time, returning to equilibrium the soonest without allowing oscillation about the equilibrium.

10) For this application, c_c, as described above offers the optimal solution, avoiding ressonance, oscillation while returning to the equilibrium position the fastest, whilst altogether providing the "softest" impact given the preceding criteria. ∴

$$c^2 = 4mk$$

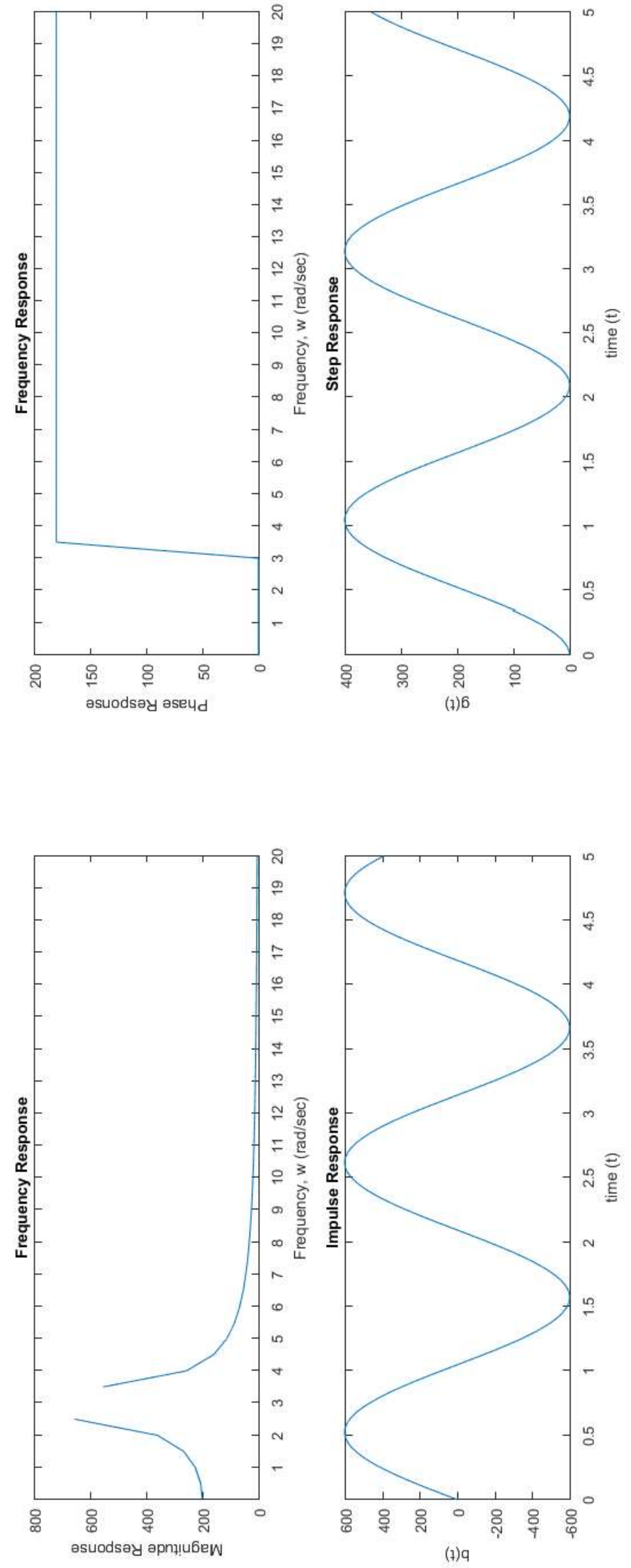
$$c = \sqrt{1440000}$$

$$= 1200 \ kg\,s^{-1}$$

11.) The transfer function, $G(s) = \frac{H(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k}$

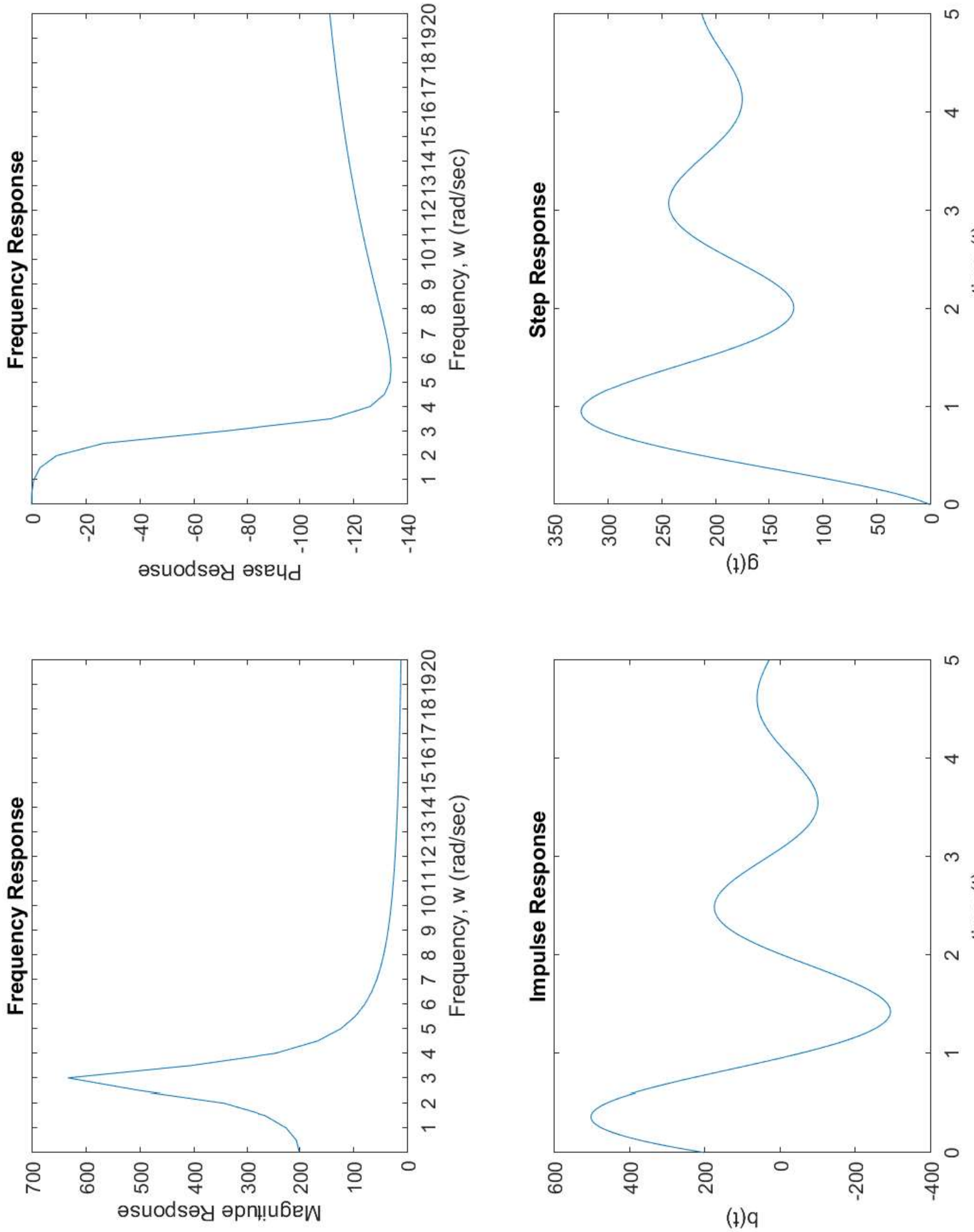
Appendix A

Undamped Case:

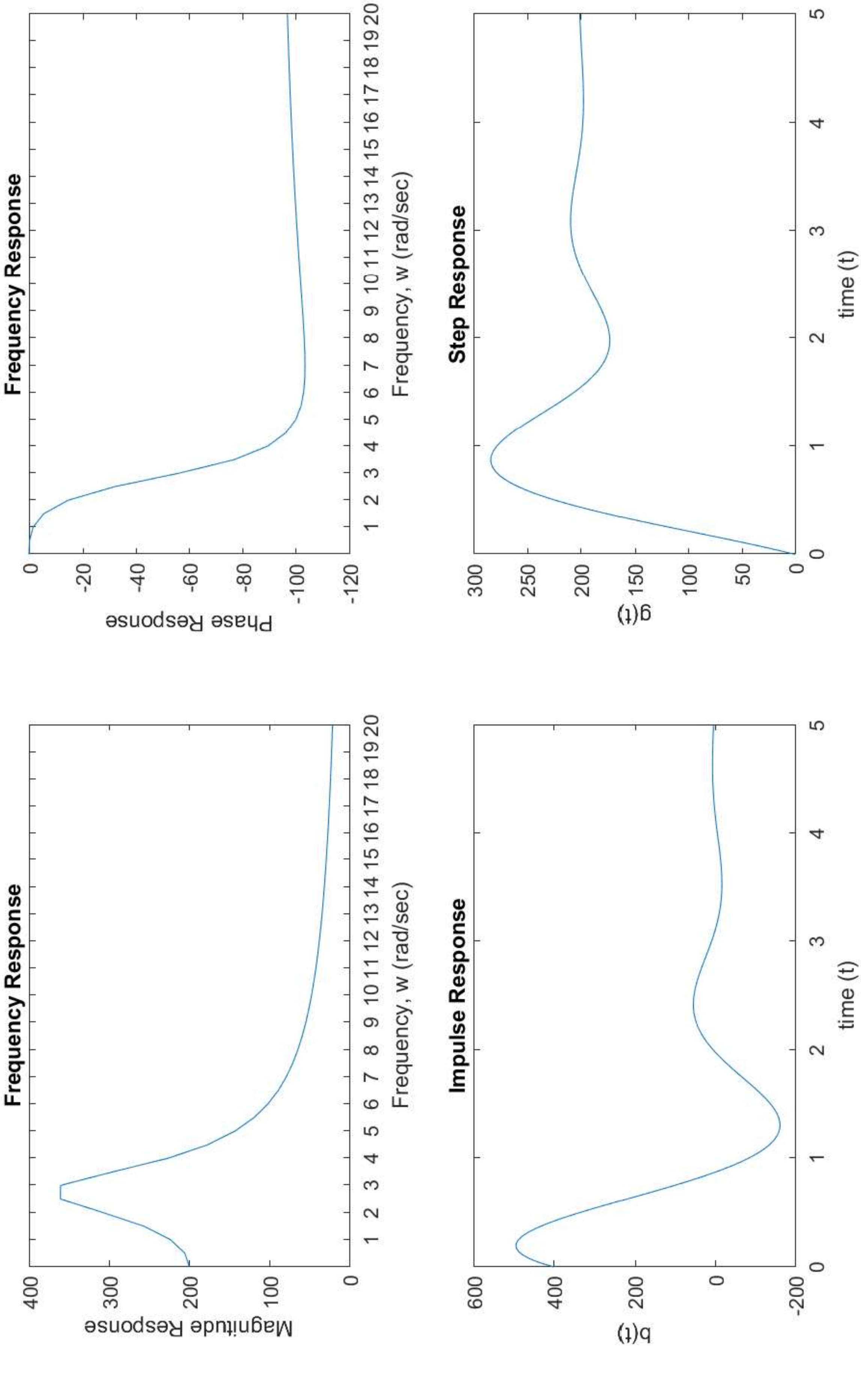


Appendix B:

Damped Case - c=200:



Damped Case - c=400:



Damped Case - c=1800: