MATH 2R03: LINEAR ALGEBRA II
HOMEWORK 3
DUE: WEDSNESDAY, 26 SEPTEMBER 2012, AT 5:00PM

Required problems (to be handed in):

1. Consider the vector space $\mathbb{R}^2$ with the usual inner product (i.e. the standard Euclidean dot product). Let θ be any real number.
   (a) Prove that, with respect to the standard dot product, the vectors $v_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, v_2 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$ form an orthonormal basis for $\mathbb{R}^2$.
   (b) Let $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Find a 2×2 real matrix A so that the function $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication by A,
   $$T_A(v) := Av,$$
   sends $T_A(u_1) = v_1$ and $T_A(u_2) = v_2$. Explain/justify your work.

2. Let $P_2[-1,1]$ be the vector space of polynomial functions on the interval $[-1,1]$ and having degree at most 2. (So, for example, the function $f(x) = x^2$ is an element of $P_2[-1,1]$, but $g(x) = \sin(x)$ is not.) Define an inner product on $P_2[-1,1]$ by the formula
   $$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx.$$
   (a) The set of functions $\mathcal{B} = \{1, x, x^2\}$ forms a basis for the vector space $P_2[-1,1]$. Use the Gram-Schmidt process to construct an orthonormal basis for $P_2[-1,1]$, starting from the basis $\mathcal{B}$. Explain/justify your work.
   (b) Now compute $\langle f, g \rangle$ where $f(x) = 1 + x$ and $g(x) = x^2 - 2x$ in $P_2[-1,1]$. Do the computation *without using any integration.* Explain/justify your work.

3. An $n \times n$ real matrix is called **orthogonal** if its column vectors are orthonormal in $\mathbb{R}^n$ (with respect to the standard Euclidean dot product).
   (a) Prove: if an $n \times n$ matrix A is orthogonal, then the columns form an orthonormal basis for $\mathbb{R}^n$ (with respect to the standard Euclidean dot product).
   (b) Prove: if P is a matrix whose columns form an orthogonal basis for $\mathbb{R}^n$, then $P^T P$ is a diagonal matrix.
   (c) (Not to hand in, but something to think about:) If P is an orthogonal matrix, what is $P^T P$?

Recommended problems (not to be handed in):

1. Let A be an orthogonal $n \times n$ matrix. Let $\{v_1, \ldots, v_n\}$ be an orthogonal basis for $\mathbb{R}^n$. Prove: $\{Av_1, Av_2, \ldots, Av_n\}$ is also an orthogonal basis for $\mathbb{R}^n$.

2. Prove that if A and B are orthogonal matrices, then so is the product AB.

3. Find an explicit, concrete example of a pair of matrices A and B such that the column vectors of both A and B are orthogonal, but such that the column vectors of AB (the matrix product) are *not* orthogonal.