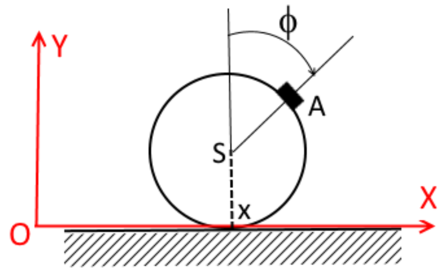

Рис. 1. ...

Imagine a hoop with mass M and radius R that will only roll without slipping on the floor. Place a point object with mass m on top of the hoop and then the system starts from at rest. Question: where does m leave M?

Reference https://www.physicsforums.com/threads/came-up-with-a-problem-that-i-cant-solve.871352/

The system is described by the following generalised coordinates: x is the coordinate of hoop's centre of mass S and ϕ is the angle between axis Y and the vector $\boldsymbol{SA}$. Here OXY is an inertial frame.

One obviously has $\boldsymbol{OA} = (x + R\sin\phi)\boldsymbol{e}_x + R\cos\phi\boldsymbol{e}_y$. And so

$$\boldsymbol{v}_A = (\dot x + R\dot\phi\cos\phi)\boldsymbol{e}_x - R\dot\phi\sin\phi\boldsymbol{e}_y.$$

The kinetic energy of the system is

$$T = \frac{1}{2}M\dot x^2 + \frac{1}{2}MR^2\Big(\frac{\dot x}{R}\Big)^2 + \frac{1}{2}m|\boldsymbol{v}_A|^2;$$

After some calculation

$$T = \Big(\frac{m}{2} + M\Big)\dot x^2 + \frac{1}{2}mR^2\dot\phi^2 + mR\dot x\dot\phi\cos\phi.$$

The gravity potential is $V = MgR\cos\phi$.

The energy integral:

$$T + V = h. \tag{1}$$

From the initial condition one has $h = MgR$.

The Lagrangian $L = T - V$ does not depend on x, consequently the system has another first integral

$$p = \frac{\partial L}{\partial \dot x} = (m + 2M)\dot x + mR\dot\phi\cos\phi. \tag{2}$$

From the initial conditions it follows that $p = 0$.

Taking into account initial conditions, integrate formula (2):

$$(m + 2M)x + mR\sin\phi = 0. \tag{3}$$

From formulas (1) and (2) we can express $\dot\phi$ as a function of ϕ : $\quad \dot\phi^2 = f(\phi)$.

Observe that by the 2nd Newton Law

$$m\dot{\boldsymbol{v}}_A = \boldsymbol{N} + m\boldsymbol{g}.$$

When the point leaves the hoop the reaction force $\boldsymbol{N} = 0$ and the 2nd Newton Law implies

$$R(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi) = g, \quad \ddot{x} + R\ddot{\phi}\cos\phi - R\dot{\phi}^2\sin\phi = 0.$$

Combining all these formulas we get equation for angle of leaving ϕ.