Exercise 1.

Proof. Conservation of angular momentum for steady state, inviscid, frictionless flow and constant density neglecting air resistance among one of three sprinkler wings with angular velocity ω_1 implies

$$\begin{aligned}
\mathbf{0} &= \iint_S (\boldsymbol{r}\times\boldsymbol{u})\boldsymbol{u}_{rel}\cdot\boldsymbol{n}\,dS && (1)\\
&= \iint_S \left(0\boldsymbol{e}_y \times \frac{Q}{3A}\boldsymbol{e}_z\right)\frac{Q}{3A}\boldsymbol{e}_z\cdot(-\boldsymbol{e}_z)\,dS + ...\\
&... + \iint_S r\boldsymbol{e}_y \times \left(-r\omega_1\boldsymbol{e}_x + \frac{Q}{3A}(\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y)\right)\frac{Q}{3A}(\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y)\cdot(\cos\theta\boldsymbol{e}_x + \sin\theta\boldsymbol{e}_y)\,dS && (2)\\
&= \frac{Q}{3A}\left(r^2\omega_1 - \frac{Qr}{3A}\cos\theta\right)\boldsymbol{e}_z\int_0^{2\pi}\int_0^{r_{pipe}} r_{pipe}\,dr_{pipe}\,d\theta \implies && (3)\\
\omega_1 &= \frac{Q\cos\theta}{3Ar} : A \equiv \int_0^{r_{pipe}} r_{pipe}\,dr_{pipe}\,d\theta. && (4)
\end{aligned}$$

However, since there are three radii, the surface integrals applied on the entire sprinkler system imply summation over three wings, and thus $\omega = \omega_1 + \omega_2 + \omega_3 \implies \boxed{\omega = Q\cos\theta/Ar}$. □

Exercise 2.

Proof. Conservation of momentum given the aforementioned assumptions implies

$$\begin{aligned}
\boldsymbol{F}_{thrust} &= m_{CV}\frac{d\boldsymbol{u}_{CV}}{dt} + \frac{\partial}{\partial t}\iiint_V \rho\boldsymbol{u}_{rel}\,dV + \iint_{\partial V}\rho\boldsymbol{u}_{rel}(\boldsymbol{u}_{rel}\cdot\boldsymbol{n})\,dS \implies && (5)\\
F_{thrust} &= m_{CV}\frac{dV(t)}{dt} + \frac{\partial}{\partial t}\iiint_V \rho(V_j - V(t))\,dV + \iint_{\partial V}(V_j - V(t))((V_j - V(t))\boldsymbol{e}_x\cdot -\boldsymbol{e}_x)\,dS +\\
&\quad + \iint_{\partial V} -(V_j - V(t))(-(V_j - V(t))\boldsymbol{e}_x\cdot -\boldsymbol{e}_x)\,dS \implies && (6)\\
F_{thrust} &= m_{CV}\frac{dV(t)}{dt} + \frac{\partial}{\partial t}m_{CV}(V_j - V(t)) - 2\rho(V_j - V(t))^2A_j \implies && (7)\\
F_{thrust} &= -2\rho(V_j - V(t))^2A_j. && (8)
\end{aligned}$$

□

Exercise 3.

Proof. □

Exercise 4.

Proof. Conservation of momentum, neglecting gravity and buoyancy implies

$$\frac{\partial}{\partial t}\iiint_V \rho\boldsymbol{u}\,dV = -\iint_{\partial V}\rho\boldsymbol{u}(\boldsymbol{u}_{rel}\cdot\boldsymbol{n})\,dS - \boldsymbol{F}_{drag} + \boldsymbol{F}_{thrust} \tag{9}$$

which, when considering only the direction of translation implies

$$\frac{\partial}{\partial t}\iiint_V [\rho_f x'(t)\,dV_f + \rho_w x'(t)\,dV_w] = -\iint_{\partial V}\rho_w x'(t)(-V_j\hat{j}\cdot(-\hat{j}))\,dS + \\ -(C_d A_s x'(t)^2 + \alpha\rho_w(V_f+V_w)) + F_{thrust} \implies \tag{10}$$

$$(m_w(t)+m_f)x''(t) + V_w'(t)\rho_w x'(t) = -A_j\rho_w x'(t)V_j - (C_d A_s x'(t)^2 + \alpha\rho_w(V_f+V_w)) + \dot{m}_w V_j \implies \tag{11}$$

$$(m_w(t)+m_f)x''(t) = -(C_d A_s x'(t)^2 + \alpha\rho_w(V_f+V_w)) + \rho_w A_j V_j^2 \tag{12}$$

subject to $x'(0) = x(0) = 0$. Notice continuity was invoked: $V_w'(t) = -V_j A_j$. $\square$