Use of Integer Arithmetic for Precise Calculation of Irrational Numbers

By Michael Druckman

This work is inspired by “A History of PI”, authored 1971 by Petr Beckmann

Beckmann speaks of the methods used to get ever more accurate calculations of the value of π resulting in the ultimate use of computers to get pages of digits of accuracy. The specific equations named were attributed to John Machin and Leonhard Euler. Solving the original equations for π the calculations can be expressed as:

John Machin

$$\pi = 4 \cdot (4 \cdot \mathrm{atan}(\frac{1}{5}) - \mathrm{atan}(\frac{1}{239}))$$

Leonhard Euler

$$\pi = 20 \cdot \mathrm{atan}(\frac{1}{7}) + 8 \cdot \mathrm{atan}(\frac{3}{79})$$

Beckmann further points out that the James Gregory formula for atan was used:

```
atan x = x - x^3/3 + x^5/5 - x^7/7 + ...
```

which for purposes of these efforts was coded as:

$$\sum_{i=1}^{n\Delta 4} \frac{x^i}{i} - \frac{x^{i+2}}{i+2}$$

Where n = 5, 9, 13, 17, 21, … depending on the number of terms desired

The script used was coded as:

```
// James Gregory atan series
// atan x = x - x^3/3 + x^5/5 - x^7/7 + ...

!!atn(x,n) = SIGMA [1 <= i <= n <> 4] (x^i/i - x^(i+2)/(i+2))
polyTerms = 10; polyDegree = polyTerms * 2 - 1; countMax = polyDegree - 2
RENDERF atn

// John Machin
//  atan (1/239) = 4 atan (1/5) - pi/4
//  pi = 4 * ( 4 * atan (1/5) - atan (1/239) )
//     = 16 * 0.19739555984988- 4 * 0.004184076002

piComputedJM = 16*atn(1/5,countMax) - 4*atn(1/239,countMax)
PRETTYPRINT piComputedJM

// Leonhard Euler
//  pi = 20 * atan (1/7) + 8 * atan (3/79)
//     = 20 * 0.1418970546 + 8 * 0.037956445188

piComputedLE = 20*atn(1/7,countMax) + 8*atn(3/79,countMax)
PRETTYPRINT piComputedLE
```

Successive executions were run with "`polyTerms`" set to varied counts to observe changes in result precisions as the correctly calculated number of decimal places increases with the count of terms.

To allow arbitrarily long series of terms without loss by decimal place truncation the calculations must be done with integer arithmetic. Since the atan function parameters are integer fractions and by use of the Gregory series which introduces no irrational function results into sub-expressions, the running series evaluation can be kept as an integer ratio. The numerator and denominator values of these ratios grow very large very rapidly. To control this complexity numerator and denominator values can be reduced by canceling common factors. This can be accomplished by prime factorization of numerator and denominator followed by recognition and removal of common factors. This has an additional side effect of keeping the series sub-expressions expressed as prime factor ratios. The results of running this code for 10 terms of each of the Machin and Euler series are listed starting on the next page

Using Machin's version [pi = 16 * atan (1/5) - 4 * atan (1/239)]

// 4 terms (6 decimal places, 1.5 per term)

3.14159{17721821773} =

76528487109180192540976 / 24359780855939418203125 =

(2^4 * 47 * 127 * 801311851955731619) / (5^7 * 7 * 239^7)

// 6 terms (9 decimal places, 1.5 per term

3.14159265{26153086} =

5150018754384552205994973405622666696 / 1639301884061026141391921953564453125 =

(2^3 * 643752344298069025749371675702833337) / (3 * 5^11 * 7 * 11 * 239^11)

// 8 terms (12 decimal places, 1.5 per term)

3.14159265358{86025} =

68264323149440668079381586907368499287891009074416 / 21729208932120203578433081040606274777175903320312 5 =

(2^5 * 89 * 127 * 151 * 124989386186520303610462356934791712845 77171) / (3 * 5^16 * 7 * 11 * 13 * 239^15)

// 10 terms (15 decimal places, 1.5 per term)

3.14159265358979{2} =

899286197155536297279342607251940330683169516449531712999212 99656

/ 2862516870632328375963019589154065793329766684818784713745117 1875 =

(2^3 * 101 * 44171 * 25197028973232524354588148961695566428355549249572927 56367)

/ (3 * 5^19 * 7 * 11 * 13 * 17 * 19 * 239^19)

Using Euler's version [pi = 20 * atan (1/7) + 8 * atan (3/79)]

```
// 4 terms (8 decimal places, 2 per term)

        3.141592{5994248215} =

        5216930888285786308592 / 1660600705909775840385 =

        ( 2^4 * 19 * 43 * 4909 * 81298028155979 ) / ( 3 * 5 * 7^8 * 79^7 )

// 6 terms (11 decimal places, 1.83 per term)

        3.1415926535{741905} =

        230001948894901784511473239934088 / 732118941751498324731500616993015 =

        ( 2^3 * 409 * 28871 * 24347613231628122587910149 ) / ( 3^2 * 5 * 7^11 * 11 * 79^11 )

// 8 terms (14 decimal places, 1.75 per term)

        3.1415926535897{882} =

        2796246113663689908874883652440123836498197152 / 890072782182157535374330602522635376186402795 =

        ( 2^5 * 41 * 2131285147609519747618051564359850485135821 ) / ( 3^2 * 5 * 7^15 * 11 * 13 * 79^15 )

// 10 terms (16 decimal places, 1.6 per term)

        3.141592653589793 =

        4445543253238546227777822036946171978625807664697905766984

                / 1415060366963480161186573164611560053913785826145542814715 =

        ( 2^3 * 1049 * 50101 * 10573358815284988545677821451160010398218210427277 )

                / ( 3^2 * 5 * 7^19 * 11 * 13 * 17 * 79^19 )
```

The numerators continue to accumulate very large prime numbers even after the common product cancelations. Beckmann speaks of past researchers looking for patterns of the digits of irrational numbers in order to find ways to more efficiently express the formulation of the value. In this endeavor the concept seems to translate into a hope that the sub-expression ratios would produce larger numbers of common product cancelations allowing for a more terse form of the display of the result.