Derivation of eqn. 22 and eqn. 23 from eqn. 21 in " Statistcs of Stokes variables for correlated Gaussian fields":

$$\begin{aligned}P(\xi_1,\xi_2,\xi_3) =&\frac{1}{4\pi}\delta(1-\sqrt{\xi_1^2+\xi_2^2+\xi_3^2})\times\frac{1-\eta_1^2-\eta_2^2-\eta_3^2}{(1-\eta_1\xi_1-\eta_2\xi_2-\eta_3\xi_3)^2} \\ =&c\cdot\frac{\delta(1-\sqrt{\xi_1^2+\xi_2^2+\xi_3^2})}{(1-\eta_1\xi_1-\eta_2\xi_2-\eta_3\xi_3)^2}\end{aligned} \tag{1}$$

where $c=(1-\eta_1^2-\eta_2^2-\eta_3^2)/(4\pi)$ and $f(\xi_1,\xi_2,\xi_3)=.$
Let $x=\xi_3$, $a=\xi_1^2+\xi_2^2$, and $y=\sqrt{a+x^2}$

$$\begin{aligned}P(\xi_1,\xi_2) =& \int_{\mathbb{R}} P(\xi_1,\xi_2,x)dx = c\int_{\mathbb{R}}\frac{\delta(1-\sqrt{a+x^2})}{(1-\eta_1\xi_1-\eta_2\xi_2-\eta_3 x)^2}dx \\ =&c\int_0^\infty \delta(1-y)\Big[\frac{1}{(1-\eta_2\xi_1-\eta_2\xi_2-\eta_3\sqrt{y^2-a})^2}+ \\ &\frac{1}{(1-\eta_2\xi_1-\eta_2\xi_2+\eta_3\sqrt{y^2-a})^2}\Big]\cdot\frac{y}{\sqrt{y^2-a}}dy \\ =&\frac{c}{\sqrt{1-\xi_1^2-\xi_2^2}}\cdot\Big[\frac{1}{(1-\eta_2\xi_1-\eta_2\xi_2-\eta_3\sqrt{1-\xi_1^2+\xi_2^2})^2}+ \\ &\frac{1}{(1-\eta_2\xi_1-\eta_2\xi_2+\eta_3\sqrt{1-\xi_1^2+\xi_2^2})^2}\Big]\end{aligned} \tag{2}$$

What's more, Let $\alpha=\eta_2$ and $\beta=\eta_3$

$$P(\xi_1)=\int_{\mathbb{R}^2}P(\xi_1,x,y)dxdy = c\int_{\mathbb{R}^2}\frac{\delta(1-\sqrt{\xi_1^2+x^2+y^2})}{(1-\eta_1\xi_1-\alpha x-\beta y)^2}dxdy = cL \tag{3}$$

Let

$$u=\frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}},\qquad v=\frac{\beta x-\alpha y}{\sqrt{\alpha^2+\beta^2}} \tag{4}$$

Then

$$L=\int_{\mathbb{R}^2}\frac{\delta(1-\sqrt{\xi_1^2+x^2+y^2})}{(1-\eta_1\xi_1-\alpha x-\beta y)^2}dxdy=\int_{\mathbb{R}^2}\frac{\delta(1-\sqrt{\xi_1^2+u^2+v^2})}{(1-\eta_1\xi_1-\sqrt{\alpha^2+\beta^2}u)^2}dudv \tag{5}$$

$$=\int_{\mathbb{R}^2}\frac{\delta(1-\sqrt{\xi_1^2+u^2+b^2})}{(A-Bu)^2}dudv=\int_{\mathbb{R}}\frac{1}{(A-Bu)^2}\Big(\int_{\mathbb{R}}\delta(1-\sqrt{\xi_1^2+u^2+v^2})dv\Big)du \tag{6}$$

Since

$$\int_{\mathbb{R}} \delta(1-\sqrt{\xi_1^2+u^2+v^2})dv = \frac{2\Theta(1-\xi_1^2-u^2)}{\sqrt{1-\xi_1^2-u^2}} \tag{7}$$

Then

$$L = \int_{\mathbb{R}} \frac{2\Theta(1-\xi_1^2-u^2)}{(A-Bu)^2\sqrt{1-\xi_1^2-u^2}}du \tag{8}$$

$$= \int_{u^2\le 1-\xi_1^2} \frac{2\Theta(1-\xi_1^2-u^2)}{(A-Bu)^2\sqrt{1-\xi_1^2-u^2}}du \tag{9}$$

$$= \int_{u^2\le r^2} \frac{2}{(A-Bu)^2\sqrt{r^2-u^2}}du \qquad (r=\sqrt{1-\xi_1^2}) \tag{10}$$

where $A = 1-\eta_1\xi_1,\ \alpha = \eta_2,\ \beta = \eta_3,\ B = \sqrt{\alpha^2+\beta^2} = \sqrt{\eta_2^2+\eta_3^2}$, and $r = \sqrt{1-\xi_1^2}$. Let $u = r\sin x$

$$L = \int_{u^2\le r^2} \frac{2}{(A-Bu)^2\sqrt{r^2-u^2}}du = \int_{-\pi/2}^{\pi/2} \frac{2}{(A-Br\sin x)^2}dx \tag{11}$$

According to

$$\int \frac{1}{(p+q\sin ax)^2} = \frac{q\cos ax}{a(p^2-q^2)(p+q\sin ax)} + \frac{p}{p^2-q^2}\int \frac{1}{p+q\sin ax}dx$$

$$\int \frac{1}{p+q\sin ax}dx = \frac{2}{a\sqrt{p^2-q^2}}\tan^{-1}(\frac{p\tan(\frac{ax}{2})+q}{\sqrt{p^2-q^2}})$$

and

$$\arctan A + \arctan B = \arctan\left(\frac{A+B}{1-AB}\right)$$

$$\arctan A - \arctan B = \arctan\left(\frac{A-B}{1+AB}\right)$$

$$\int_{-\pi/2}^{\pi/2} \frac{1}{(A-Br\sin x)^2}dx = \frac{-Br\cos x}{(A^2-B^2r^2)(A-Br\sin x)}\Big|_{-\pi/2}^{\pi/2} + \frac{A}{A^2-B^2r^2}\int_{-\pi/2}^{\pi/2} \frac{1}{A-Br\sin x}dx$$

$$= \frac{A}{A^2-B^2r^2}\Big[\frac{2}{\sqrt{A^2-B^2r^2}}\Big(\tan^{-1}(\frac{A-Br}{A^2-B^2r^2}) - \tan^{-1}(\frac{-A-Br}{A^2-B^2r^2})\Big)\Big]$$

$$= \frac{2A}{(A^2-B^2r^2)^{3/2}}\tan^{-1}(\infty) = \frac{2A}{(A^2-B^2r^2)^{3/2}}\cdot\frac{\pi}{2} = \frac{A\pi}{(A^2-B^2r^2)^{3/2}} \tag{12}$$

Substituting eqn. (12) into eqn. (3), we have

$$f(\xi_1) = c \cdot 2 \cdot \frac{A\pi}{(A^2 - B^2 r^2)^{3/2}} = \frac{(1-\eta_1\xi_1)(1-\eta_1^2-\eta_2^2-\eta_3^2)}{2[(1-\eta_1\xi_1)^2 - (\eta_2^2-\eta_3^2)(1-\xi_1^2)]^{3/2}} \times \Theta(1-|\xi_1|).$$