Intro/Summary of Integration

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1 Definite Integrals

A small amount of knowledge about limits will be required for this first section as we introduce a formal definition of the definite integral.

We define the definite integral of a real valued continuous function $f(x)$ on the interval $[a, b]$ to be,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x,$$

where δx are equal subintervals of $[a, b]$ and x_i are sampling points of each subinterval.

Here we have chosen to define the subintervals to be equal in length for convenience. We call $f(x)$ the integrand, a and b the limits of integration, and $[a, b]$ the interval of integration.

Now we have eliminated some of the terminology, let us look at where this definition comes from. The definiton comes from efforts in the past to try and find areas under curves. If we consider a curve defined by $f(x)$ that is continuous and real-valued, then how can we find the area between the curve, the x-axis and bounded by $x = a$ and $x = b$? As you may have guessed from the definition, we can divide the interval $[a, b]$ into n smaller subintervals and approximate the area between the curve and the x-axis in the subinterval as that of a rectangle. To do this we need to find the height of the rectangle which we obtain from choosing a sampling point, x_i, of the ith subinterval.

In *figure 1* we can see a curve that has the interval $[a, b]$ split into several subintervals. In general we can split the interval into n subintervals each of length $\delta x = \frac{b-a}{n}$. We expect that the more subintervals we have the better the approximation of area will be and in the limit that $n \to \infty$, we will have calculated the area under the curve. Thus we can write,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x.$$

We note from the figure that the sampling point x_i is chosen to be the end of each subinterval for ease of calculation. Let us now look at an example to see this method in action.

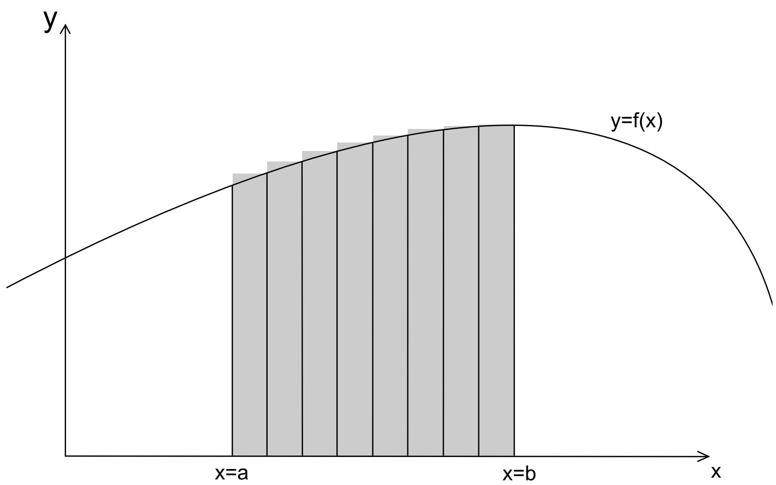

Figure 1: Approximating the area under a curve.

Example 1.1

Find $\int_0^2 x dx$

We note that the area can easily be worked out as it is simply the area of a triangle. The first thing we do is divide the interval of integration into n subintervals: $I_1, I_2, ..., I_n$.

$$I_i = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right].$$

This corresponds to an interval length of $\delta x = \frac{2-0}{n} = \frac{2}{n}$. We choose $x_i = \frac{2i}{n}$ which is the end of the interval I_i. Thus we have $f(x_i) = \frac{2i}{n}$. The total area of the region is then:

$$A = \sum_{i=1}^{n} f(x_i)\delta x = \sum_{i=1}^{n} \frac{2i}{n}\frac{2}{n} = \frac{4}{n^2}\sum_{i=1}^{n} i$$

We note that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, and thus we have.

$$A = \frac{4(n^2+n)}{2n^2} = 2 + \frac{2}{n}$$

Now taking the limit as $n \to \infty$ we finally get:

$$A = 2$$

That was a relatively easy example but you could try it with x^2 or other functions if you so wish. Even though the answer was obvious we had a lot of work to do just to get it. We would also have a lot of work to do whatever function we choose and this brings us to the next section.

2 The Fundamental Theorem of Calculus

Here we introduce the Fundamental Theorem of Calculus. The theorem is usually expressed in two parts and a brief explanation of what each part means will follow. Often they are introduced in a different order in some texts and resources so don't worry if you come across that.

If F is defined by,

$$F(x) = \int_a^x f(t)dt$$

then,

$$F'(x) = f(x)$$

This neat result shows us that integration is the inverse process of differentiation.

If f is a continuous, real valued function on the interval $[a, b]$, and F is the indefinite integral (sometimes called the anti-derivative or primitive) of f then,

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b.$$

This just relates the indefinite integral to the definite integral. This means that all we have to do now to evaluate an integral is look for a function $F(x)$ that differentiates to become the integrand, $f(x)$.

I will not provide a proof of either of these two theorems as they are dealt with exstensively elsewhere. I think the visual calculus web site has one of the best walkthroughs of the proof available on the web. One can access them from *this* [1] page.

[1] http://archives.math.utk.edu/visual.calculus/4/ftc.9/

3 Rules of integration

I will first list some standard integrals which are general results of indefinite integration of some simple functions. Then I will list the methods used for dealing with combinations of these functions.

3.1 Standard Integrals

In differentiation we remember the general results of differentiating simple functions and we also remember rules that allow us to differentiate combinations of these simple functions. With integrals we can do the same thing. Now we know that the primitive is simply a function we can differentiate to get the integrand we can work out some general results for simple functions quite easily. The following is a list of some standard integrals you may come across.

- $\displaystyle\int k dx = kx + c$
- $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\displaystyle\int \frac{1}{x} dx = \ln|x| + c$
- $\displaystyle\int \sin(x) dx = -\cos(x) + c$
- $\displaystyle\int \cos(x) dx = \sin(x) + c$
- $\displaystyle\int \tan(x) dx = -\ln|\cos(x)| + c$
- $\displaystyle\int e^x dx = e^x + c$

You will notice the inclusion of an arbitrary constant, c, in the above list of indefinite integrals. Just a brief note on why this is so. We know from differential calculus that the derivative of a constant is zero. Therefore if $F(x)$ is the indefinite integral of $f(x)$, then so is $F(x) + c$ since $\frac{d}{dx}(F(x) + c) = \frac{d}{dx}F(x) + \frac{d}{dx}c = f(x) + 0 = f(x)$.

This is a brief list of the most common functions, but more detailed lists can be found in most text books dealing with integration. I suggest that you find one that you like and know where to find it if needed.

3.2 Properties of Integrals

3.2.1 Integrating a sum of functions

Suppose from the fundamental theorem that $F'(x) = f(x)$ and $G'(x) = g(x)$, then:

From the derivative of a sum we have:

$$(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x)$$
$$\boxed{\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx}$$

3.2.2 Integrating a Constant Multiple of a Function

For a function F that is the primitive of f and again borrowing some rules from differentiation:

$$kF'(x) = kf(x)$$
$$\boxed{\Rightarrow \int kf(x)dx = k \int f(x)dx}$$

3.2.3 Swapping Limits

Going right back to our definition of a definite integral we find that:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\delta x$$

Where we had $\delta x = \frac{b-a}{n}$. It then follows that:

$$\int_b^a f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta x$$

Where this time $\delta x = \frac{a-b}{n}$. Now:

$$\begin{aligned}
\int_a^b f(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(-\frac{a-b}{n}\right) \\
&\boxed{= -\int_b^a f(x)dx}
\end{aligned}$$

3.2.4 Splitting Limits

Again we must use our definition of a definite integral.

$$\begin{aligned}
\int_a^c f(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{c-a}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}+\frac{c-b}{n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) + \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{c-b}{n}\right) \\
&\boxed{= \int_a^b f(x)dx + \int_b^c f(x)dx}
\end{aligned}$$

4 Integration by Substitution

The goal of integration by substitution is to turn a difficult integral into an easier one through a suitable substitution.

Example 4.1

Consider the following indefinite integral.

$$\int \frac{x^2 dx}{x^3 + 1}$$

We notice that the integral can be simplified if we use the following substitution:

$$u = x^3 + 1 \Rightarrow du = 3x^2 dx$$

Then,

$$\begin{aligned} \int \frac{x^2 dx}{x^3 + 1} &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln |u| + c \\ &= \frac{1}{3} \ln |x^3 + 1| + c \end{aligned}$$

We turned this difficult integral into one that was more managable by a carefully chosen substitution. The general form of integration by substitution stems from the chain rule of differentiation. Thus for a function F which is the anti-derivative of f so $F'(u) = f(u)$ we have,

$$\frac{d}{dx} F(g(x)) = \frac{d}{du} F(u) \frac{du}{dx} = f(u) g'(x) = f(g(x)) g'(x)$$

Therefore,

$$\boxed{\int f(g(x)) \frac{du}{dx} dx = \int f(u) du}. \tag{1}$$

Here we have assumed that $u = g(x)$. Students new to the method of integration by substitution often wonder how one can tell that a particular problem can be solved by this method. The short answer is that it takes a lot of practice and sometimes a lot of trial and error. If you do enough practice problems however, you will soon start to recognise which integrals are suitable for the method of substitution and which ones are not.

4.1 Trigonometric Substitutions

In some integrals we can make a trigonometric substitution that will reduce the integral to a simpler form. Lets look at an example to see how it works.

Example 4.1.1

Find $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ for some real constant a.

So we want to set x to a function $g(u)$ that will simplify the integrand enough to perform the integration. We will choose $x = a\sin(u)$ for reasons that will hopefully become very clear as we move through the problem, but which I will state explicitly later on. So with this choice for x we have:

$$\frac{dx}{du} = a\cos(u)$$

Thus:

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a\cos(u)du}{\sqrt{a^2 - a^2\sin^2(u)}} \\
&= \int \frac{a\cos(u)du}{a\cos(u)} \\
&= \int du \\
&= u + c \\
&= sin^{-1}\left(\frac{x}{a}\right) + c
\end{aligned}$$

We notice a couple of things while getting to this answer. First we notice that the substitution is not immediately obvious but once we see how the trigonometric identity ($\sin^2(u) + \cos^2(u) = 1$) changes it, then it becomes quite clear. Secondly we note that to get the answer in terms of x, $g(u)$ must have an inverse function, which in this case it did.

There are several standard substitutions where the integrand contains a factor similar to that in the example. They are listed below.

Factor	Substitution (for x)
$\sqrt{a^2 - x^2}$	$a\sin(u)$ or $a\,\mathrm{sech}(u)$
$\sqrt{a^2 + x^2}$	$a\tan(u)$ or $a\sinh(u)$
$\sqrt{x^2 - a^2}$	$a\cosh(u)$ or $a\sec(u)$

We choose these substitutions because of their corresponding trigonometric and hyperbolic identities which more often than not simplify the integrand in a similar fashion to the example above.

5 Integration by Parts

Just like integration by substitution is associated with the chain rule of differentiation, integration by parts is associated with the product rule of differentiation. Lets remind ourselves of what the product rule is:

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx} \tag{2}$$

If we integrate equation (2) we obtain,

$$f(x)g(x) = \int f(x)\frac{dg}{dx}dx + \int g(x)\frac{df}{dx}dx,$$

and rearranging we finally get.

$$\int f(x)\frac{dg}{dx}dx = f(x)g(x) - \int g(x)\frac{df}{dx}dx$$

You will often see this written in the following shorthand way.

$$\int udv = uv - \int vdu$$

where $u = f(x)$ and $v = g(x)$. We now look at an example to see this in action.

Example 5.1

Find $\int xe^x dx$

We choose $u = x$ and $dv = e^x dx$. That gives us, $du = dx$ and $v = e^x$. Now,

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c\end{aligned}$$

You can check that this is correct by differentiating it with respect to x.

5.1 Applications of Integration by Parts

5.1.1 Reduction Formula

Ocassionally one will have to apply the method of integration by parts several times before arriving at a soultion. This can be tiresome but also unnecessary because in many cases a reduction formula can be found that will speed up the process. Take for example the following integral.

Example

Find $\displaystyle\int x^3 e^x dx$

Set $u = x^3$ and $dv = e^x dx$ and then we obtain, $du = 3x^2 dx$ and $v = e^x$. Applying integration by parts:

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx$$

We notice that after we have applied the method of integration by parts that we are left with an integral on the right hand side that requires us to apply the method of integration by parts again. To make this simpler we note that:

$$\begin{aligned}\int x^n e^x dx &= x^n e^x - n \int x^{n-1} e^x dx \\ I_n &= x^n e^x - n I_{n-1}\end{aligned}$$

We can now apply this reduction formula to our example above as many times as needed.

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - 3 \int x^2 e^x dx \\ &= x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x dx \right) \\ &= x^3 e^x - 3 \left(x^2 e^x - 2 \left(x e^x - \int e^x dx \right) \right) \\ &= \left(x^3 - 3x^2 + 6x - 6 \right) e^x + c\end{aligned}$$

This has helped us because we no longer need to choose a u and dv and apply the integration by parts from scratch.

5.1.2 Integrals Involving Trigonometric Functions

Another application of integration by parts is when dealing with integrals of the form $\int e^{ax} \sin(bx) dx$ and $\int e^{ax} \cos(bx) dx$.

Lets take a look at an example.

Example

Find $\displaystyle\int e^{ax}\sin(bx)dx$

Set $u = e^{ax}$ and $dv = \sin(bx)dx$, then $du = ae^{ax}dx$ and $v = -\frac{1}{b}\cos(bx)$. Applying integration by parts:

$$\int e^{ax}\sin(bx)dx = -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\int e^{ax}\cos(bx)dx$$

We note we will have to apply integration by parts again to the integral on the right hand side. At this point you may also be thinking that we could keep doing this forever, but for now lets just apply integration by parts once more and see what we get.

Set $u = e^{ax}$ and $dv = \cos(bx)dx$, then $du = ae^{ax}dx$ and $v = \frac{1}{b}\sin(bx)$. Applying integration by parts:

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx$$

Now our original integral $\displaystyle\int e^{ax}\sin(bx)dx = I$ looks like this:

$$\begin{aligned}
I &= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b}\left(\frac{e^{ax}\sin(bx)}{b} - \frac{a}{b}\int e^{ax}\sin(bx)dx\right) \\
&= -\frac{e^{ax}\cos(bx)}{b} + \frac{a}{b^2}e^{ax}\sin(bx) - \frac{a^2}{b^2}I \\
I + \frac{a^2}{b^2}I &= \frac{a}{b^2}e^{ax}\sin(bx) - \frac{1}{b}e^{ax}\cos(bx) \\
I\left(\frac{b^2+a^2}{b^2}\right) &= \frac{a}{b^2}e^{ax}\sin(bx) - \frac{1}{b}e^{ax}\cos(bx) \\
I &= \frac{e^{ax}\left(a\sin(bx) - b\cos(bx)\right)}{b^2+a^2} + c
\end{aligned}$$

And thus we arrive at an answer for our original integral when it seemed we may get stuck applying integration by parts forever. Of course this technique is not only limited to the forms of integrals I gave above. Now the technique has been introduced and you have seen it in action it may become more apparent in other problems. In essence one just needs to recognise when you've arrived back at the original integral after several applications of integration by parts.

References

Gilbert, J. and Jordan, C., *Guide to Mathematical Methods*, Palgrave Macmillan, 2002, 2nd Edition.

Lambourne, R. and Tinker, M., *Basic Mathematics for the Physical Sciences*, Wiley, Chichester, 2001.