

An analytical treatment of the Clock Paradox in the framework of the Special and General Theories of Relativity

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Abstract

In this paper we treat the so called clock paradox in an analytical way by assuming that a constant and uniform force \mathbf{F} of *finite* magnitude acts continuously on the moving clock *along the direction of its motion* assumed to be *rectilinear* (in space). No inertial motion steps are considered. The rest clock is denoted as (1), the to-and-fro moving clock is (2), the inertial frame in which (1) is at rest in its origin and (2) is seen moving is I and, finally, the accelerated frame in which (2) is at rest in its origin and (1) moves forward and backward is A . We deal with the following questions: I) *What is the effect of the finite force acting on (2) on the proper time intervals $\Delta\tau^{(1)}$ and $\Delta\tau^{(2)}$ measured by the two clocks when they reunite? Does a differential aging between the two clocks occur*, as it happens when inertial motion and infinite values of the accelerating force is considered? The Special Theory of Relativity is used in order to describe the hyperbolic (in spacetime) motion of (2) in the frame I II) *Is this effect an absolute one*, i.e. does the accelerated observer A comoving with (2) obtain the same results as that obtained by the observer in I , both qualitatively and quantitatively, as it is expected? We use the General Theory of Relativity in order to answer this question. It turns out that $\Delta\tau_I = \Delta\tau_A$ for both the clocks, $\Delta\tau^{(1)}$ and $\Delta\tau^{(2)}$ *do depend on* $g = F/m$, and $\Delta\tau^{(2)}/\Delta\tau^{(1)} = (\sqrt{1 - \beta^2} \operatorname{atanh} \beta)/\beta < 1$. In it $\beta = V/c$ and V is the velocity acquired by (2) when the force inverts its action.

1 Introduction

In this paper we wish to quantitatively examine in detail the so called clock paradox by accounting for the effects of the *finiteness* of the force \mathbf{F} which

accelerates and decelerates the moving clock which will conventionally be denoted as (2). The Special and General Theories of Relativity will be used, as in [1] in which a symmetrical version of the clock paradox has been considered. In it two clocks perform hyperbolic motions with oppositely directed velocities and accelerations. The role of finite acceleration in the twin paradox has recently been investigated in [2, 3] without using the Einstein theory of accelerated frames.

An unidimensional *rectilinear* (in space) path will be considered in which the moving clock starts its motion with zero initial velocity from the origin of the inertial frame I in which the rest clock, denoted conventionally as (1), is located. After the velocity V is reached, \mathbf{F} is instantaneously reversed. Then, (2) is decelerated, stops and inverts its motion until the velocity $-V$ is reached. At this point \mathbf{F} is suddenly reversed again, so that, when (2) stops, it meets (1) again and they compare their readings.

In general, by assuming an infinite force (or no force at all¹) on the accelerated clock, it can be obtained that (2) does lag behind the inertial clock (1) when they reunite and compare their readings and that this asymmetric effect is an *absolute* one. The paradoxical symmetric outcome would come, instead, from an uncorrect application of the Special Theory of Relativity to the motion of (2) in the sense that the so called time ‘jump’ in the time of (1) as measured by (2) during its inertial motion must be considered as well in order to obtain the desired result. Instead, if it is not accounted for, the paradoxical symmetric situation comes out.

In this paper we wish to investigate what happens if, instead, a *finite* force is considered: *does a differential aging between the two clocks occur again? If so, what is the clock which lags behind? What is the magnitude of this effect?* If the relation between the proper times measured when they reunite again in order to compare the readings of their displays had to be considered as an absolute effect, both the inertial observer and the accelerated one should agree not only that a certain clock lags behind the other clock, but also the magnitude of such an effect should be the same. We will investigate quantitatively *if and how this feature is modified by accounting for the finite force experienced by (2) leading to some inconsistencies*. Indeed, the elapsed proper time is function only of the observer’s worldline and of its starting/ending points: it is the Lorentzian length of the segment of worldline delimited by given endpoints. The spacetime of an accelerated

¹Indeed, it is possible to imagine a situation with one observer at rest and two oppositely moving inertial observers which encounter each other at a certain spacetime event in which no acceleration occurs at all. By accounting for the time ‘jump’ it is possible to obtain the desired absolute different aging at the reunion with the rest clock [4].

frame does not present a real curvature as if a true gravitational field was present. Whatever coordinates are used for flat spacetime, it will always be flat. The coordinates used might change the form of the metric, but they cannot create curvature. The choice of frame will also change the coordinate expression of geodesics (e.g., the worldline of clock (1)), but it will not change its geometrical properties, including its proper-time length. Thus, the worldline will curve in the sense that its spatial coordinates are not constant in the accelerated frame, but it will still be straight in the sense that it is inertial (and therefore an extremum of proper-time length).

The motion will be described both in the inertial frame I , in which (1) is at rest while (2) moves according to the special relativistic hyperbolic (in spacetime) motion, and in the accelerated frame A comoving with (2) in which the latter one is constantly at rest and (1) moves in a way which will be derived in the framework of the General Theory of Relativity. In general, t will denote the proper time of the clock which is at rest in a given frame and τ the proper time of the clock which is seen to be in motion in the same frame.

What is the experimental status of the clock paradox? Some experiments with elementary particles have been carried out until now. The celebrated experience of the circling muons at CERN [5] refers to a scenario in which the moving particles, following a *circular uniform* motion, comes back periodically to the same point where their lifetimes are measured and compared to their proper lifetimes calculated at rest. Note that, in this case, *no acceleration along the direction of motion* exists as, instead, is the case of the rectilinear accelerated motion. In the case of a *rectilinear* motion there is *no* any experiment which tests a scenario like that involved in the to-and-fro journey of the *accelerated* clock; the observed dilation of the lifetime of muons in the cosmic rays which reach the sea level before decaying [6] does not imply a return of the muons to some point of reference. Moreover, no external force acts on the cosmic rays muons whose motion is rectilinear and uniform, i.e. an *one-way non-accelerated motion* occurs in this case which *cannot be assumed as an experimental test of the clock paradox in the case of rectilinear accelerated motion*.

2 The point of view of the inertial clock

2.1 The special relativistic equation of motion of an accelerated particle in an inertial frame

The equation of motion of a particle of mass m acted upon by a force \mathbf{F} , as viewed in an inertial frame I , is, according to the Special Theory of Relativity

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \quad (1)$$

with

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

Note that t denotes the proper time of a standard clock located at the origin of I .

Let us consider an unidimensional motion under the action of a constant and uniform force F . A first integration of eq.(1) yields

$$\frac{v(t)}{\sqrt{1 - \left[\frac{v(t)}{c}\right]^2}} = g(t - t_0) + C_1, \quad (2)$$

with $g \equiv F/m$. Eq.(2) admits as solution

$$x(t) = \frac{c^2}{g} \sqrt{1 + \left[\frac{g(t - t_0)}{c} + \frac{C_1}{c}\right]^2} + C_2,$$

from which the velocity can be obtained

$$\frac{dx(t)}{dt} = c \frac{\left[\frac{g(t-t_0)}{c} + \frac{C_1}{c}\right]}{\sqrt{1 + \left[\frac{g(t-t_0)}{c} + \frac{C_1}{c}\right]^2}}.$$

The constants of integration can be determined from the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$; then,

$$\begin{aligned} C_1 &= \frac{v_0}{\sqrt{1 - \left(\frac{v_0}{c}\right)^2}}, \\ C_2 &= x_0 - \frac{c^2}{g \sqrt{1 - \left(\frac{v_0}{c}\right)^2}}, \end{aligned}$$

so that the complete characterization of the special relativistic hyperbolic motion is

$$x - x_0 = \frac{c^2}{g} \left\{ \sqrt{1 + \left[\frac{g(t-t_0)}{c} + \frac{v_0}{c\sqrt{1 - (\frac{v_0}{c})^2}} \right]^2} - \frac{1}{\sqrt{1 - (\frac{v_0}{c})^2}} \right\}, \quad (3)$$

$$\frac{dx}{dt} = \frac{g(t-t_0) + \frac{v_0}{\sqrt{1 - (\frac{v_0}{c})^2}}}{\sqrt{1 + \left[\frac{g(t-t_0)}{c} + \frac{v_0}{c\sqrt{1 - (\frac{v_0}{c})^2}} \right]^2}}, \quad (4)$$

Note that, for $c \rightarrow \infty$, eqs.(3)-(4) tend to

$$\begin{aligned} x(t) &\rightarrow x_0 + v_0(t-t_0) + \frac{g(t-t_0)^2}{2}, \\ \frac{dx(t)}{dt} &\rightarrow v_0 + g(t-t_0). \end{aligned} \quad (5)$$

The proper time interval of the moving particle can be written as

$$\tau - \tau_0 = \int_{t_0}^t \sqrt{1 - \left[\frac{v(t')}{c} \right]^2} dt', \quad (6)$$

according to the hypothesis of locality [7]. It is interesting to note that eq.(6) does *not* depend *explicitly* on the *acceleration* of the moving particle, but only on its velocity which, however, contains the force per unit mass g , as can be noted by eq.(4).

2.2 The motion of the clock (2) with respect to the clock (1)

The results of the previous section will now be used in order to describe the motion of the clock (2) with respect to the clock (1). In this case t is the proper time $\tau^{(1)}$ of the clock (1), which is at rest in I , while τ denotes the proper time $\tau^{(2)}$ of the moving clock (2). In the following we will split the motion in four steps.

2.2.1 From $t = t_0$ to $t = t_1$

In this stage

- $g > 0$
- $x(t) > 0$
- $v(t) > 0$
- $x(t_0) = 0$
- $x(t_1) = \lambda_1$
- $v(t_0) = 0$
- $v(t_1) = V$

i.e. the clock (2) starts moving from the origin with zero initial velocity and is accelerated to a velocity V which is reached at $t = t_1$ in λ_1 . From eq.(2) it can be obtained

$$t_1 - t_0 = \frac{V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}}. \quad (7)$$

Note that, for finite values of g , $t_1 - t_0$ is finite as well; the larger the force acting on (2), the shorter the time required to reach V . From eq.(3) and eq.(7) it can be obtained the point at which the velocity V is reached: it is

$$\lambda_1 = \frac{c^2}{g} \left[\frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} - 1 \right] \equiv \lambda. \quad (8)$$

Note that, for $g \rightarrow \infty$, it tends to zero; for finite values of g it is finite as well².

Eq.(2) in eq.(6) yields

$$\tau_1 - \tau_0 = \frac{c}{g} \ln \left\{ \frac{g(t_1 - t_0)}{c} + \sqrt{1 + \left[\frac{g(t_1 - t_0)}{c} \right]^2} \right\}. \quad (9)$$

Finally, eq.(7) in eq.(9) allows to obtain

$$\tau_1 - \tau_0 = \frac{c}{g} \operatorname{atanh} \left(\frac{V}{c} \right). \quad (10)$$

Note that, *during the accelerated motion, the proper time interval of the moving clock (2) is always shorter than the proper time read by the inertial clock (1)*. Note that, for $g \rightarrow \infty$, both eq.(7) and eq.(10) vanish.

²The topic of lengths in accelerated frames has recently been treated in [8, 9].

2.2.2 From $t = t_1$ to $t = t_2$

In this stage

- $g < 0$
- $x(t) > 0$
- $v(t) > 0$
- $x(t_1) = \lambda$
- $x(t_2) = L$
- $v(t_1) = V$
- $v(t_2) = 0$

i.e. at t_1 the same force as before is switched again on, but in the opposite direction, so that the clock (2) decelerates and stops at t_2 in $x(t_2) = L$. From eq.(2) it can be obtained

$$t_2 - t_1 = \frac{V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}} = t_1 - t_0. \quad (11)$$

Eq.(11) in eq.(3) yields

$$L = \lambda + \frac{c^2}{g} \left[\frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} - 1 \right]. \quad (12)$$

From eq.(8) it follows

$$L = 2\lambda = \frac{2c^2}{g} \left[\frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} - 1 \right]. \quad (13)$$

The proper time interval of the clock (2) is, according to the initial conditions of this stage

$$\begin{aligned} \tau_2 - \tau_1 = & \frac{c}{2g} \ln \left\{ \left[\frac{1 + \left(\frac{V}{c}\right)}{1 - \left(\frac{V}{c}\right)} \right] \left[\frac{g(t_2 - t_1)}{c} - \frac{V}{c\sqrt{1 - \left(\frac{V}{c}\right)^2}} + \right. \right. \\ & \left. \left. + \sqrt{\frac{1}{1 - \left(\frac{V}{c}\right)^2} - \frac{2gV(t_2 - t_1)}{c^2\sqrt{1 - \left(\frac{V}{c}\right)^2}} + \frac{g^2(t_2 - t_1)^2}{c^2}} \right] \right\}. \quad (14) \end{aligned}$$

Eq.(11) in eq.(14) yields

$$\tau_2 - \tau_1 = \frac{c}{g} \operatorname{atanh} \left(\frac{V}{c} \right) = \tau_1 - \tau_0. \quad (15)$$

2.2.3 From $t = t_2$ to $t = t_3$

In this stage

- $g < 0$
- $x(t) > 0$
- $v(t) < 0$
- $x(t_2) = L$
- $x(t_3) = \lambda_2$
- $v(t_2) = 0$
- $v(t_3) = -V$

i.e. the force continues to act upon the clock (2) along the negative x axis so that it starts accelerating until velocity $-V$ is reached at t_3 . From eq.(2) it can be obtained

$$t_3 - t_2 = \frac{V}{g \sqrt{1 - \left(\frac{V}{c}\right)^2}} = t_2 - t_1 = t_1 - t_0. \quad (16)$$

Eq.(16) in eq.(3) yields for λ_2 , i.e. the place where F is reversed

$$\lambda_2 = -\frac{c^2}{g} \left[\frac{1}{\sqrt{1 - \left(\frac{V}{c}\right)^2}} - 1 \right] + L = \lambda. \quad (17)$$

Note that, for $g \rightarrow \infty$, it tends to L . Eq.(2) in eq.(6) yields

$$\tau_3 - \tau_2 = \frac{c}{g} \ln \left\{ \frac{g(t_3 - t_2)}{c} + \sqrt{1 + \left[\frac{g(t_3 - t_2)}{c} \right]^2} \right\}. \quad (18)$$

Finally, eq.(16) in eq.(18) allows to obtain

$$\tau_3 - \tau_2 = \frac{c}{g} \operatorname{atanh} \left(\frac{V}{c} \right) = \tau_2 - \tau_1 = \tau_1 - \tau_0. \quad (19)$$

2.2.4 From t_3 to t_4

In this stage

- $g > 0$
- $x(t) > 0$
- $v(t) < 0$
- $x(t_3) = \lambda$
- $x(t_4) = 0$
- $v(t_3) = -V$
- $v(t_4) = 0$

i.e. the same force as before is switched again on, but in the opposite direction, at t_3 and the clock (2) is decelerated until it stops at t_4 when it meets the clock (1) again. From eq.(2) it can be obtained

$$t_4 - t_3 = \frac{V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}} = t_3 - t_2 = t_2 - t_1 = t_1 - t_0. \quad (20)$$

The proper time interval of the clock (2) is

$$\begin{aligned} \tau_4 - \tau_3 = & \frac{c}{2g} \ln \left\{ \left[\frac{1 + \left(\frac{V}{c}\right)}{1 - \left(\frac{V}{c}\right)} \right] \left[\frac{g(t_4 - t_3)}{c} - \frac{V}{c\sqrt{1 - \left(\frac{V}{c}\right)^2}} + \right. \right. \\ & \left. \left. + \sqrt{\frac{1}{1 - \left(\frac{V}{c}\right)^2} - \frac{2gV(t_4 - t_3)}{c^2\sqrt{1 - \left(\frac{V}{c}\right)^2}} + \frac{g^2(t_4 - t_3)^2}{c^2}} \right] \right\}. \quad (21) \end{aligned}$$

Eq.(20) in eq.(21) yields

$$\tau_4 - \tau_3 = \frac{c}{g} \operatorname{atanh} \left(\frac{V}{c} \right) = \tau_3 - \tau_2 = \tau_1 - \tau_0. \quad (22)$$

2.3 The total proper time intervals at the clocks' reunion

The total proper time interval of the accelerated clock (2) at the reunion with the rest clock (1), as viewed in I , is, then

$$\Delta\tau_I^{(2)} = \frac{4c}{g} \operatorname{atanh} \left(\frac{V}{c} \right), \quad (23)$$

while the total proper time interval of the rest clock (1) at the reunion with the moving clock (2), as viewed in I , is

$$\Delta\tau_I^{(1)} = \frac{4V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}}. \quad (24)$$

Note that eqs.(23)-(24) *do depend on the magnitude of the force per unit mass applied to (2). However, also in presence of finite values of the force which acts upon the clock (2), it lags always behind the rest clock (1).*

3 The point of view of the accelerated clock

Does the observer comoving with (2) obtain the same results of (1) in I seen in the previous Section? It has been shown that it is possible to preserve the absolute differential aging of the two clocks when no accelerations at all occur (see, e.g. [4]) by using three inertial observers. What happens if a finite acceleration felt by (2) is taken into account when the point of view of (2) is considered? In this Section we will try to answer this question by using the formalism of the General Theory of Relativity.

3.1 The equations of motion in an accelerated frame

Let us consider the generic motion of a body with respect to an inertial frame whose spacetime coordinates will be now denoted as (X, Y, Z, T) . It is possible to construct a frame of reference which is the relativistic analogue of a classical rigid³ frame of Cartesian axes following the body in its motion, so that the latter is constantly situated at the origin of this frame of reference [10]. It will be denoted as A and its spacetime coordinates will be (x, y, z, t) . It can be constructed from the successive infinitesimal Lorentz

³A system of reference is called rigid if the space distance σ between two reference points, as measured by standard measuring-rods at rest in the system, is constant in time. In this case $(d\sigma)^2 = (dx)^2 + (dy)^2 + (dz)^2$ and the coordinates x, y, z are Cartesian and have a metric meaning.

transformations without rotations of the spatial axes which determine the successive inertial frames that are momentarily rest systems of the moving particle. It turns out that the coordinate clock located at the origin is a standard clock, i.e. $t = \tau'$, where τ' is the proper time of the moving body on which the frame A is constructed and which is located at its origin. For a hyperbolic motion along the x axis the transformation between the inertial frame and A is

$$X = \frac{c^2}{g} \left(\cosh \frac{gt}{c} - 1 \right) + x \cosh \frac{gt}{c}, \quad (25)$$

$$Y = y, \quad Z = z, \quad (26)$$

$$T = \frac{c}{g} \sinh \frac{gt}{c} + \frac{x}{c} \sinh \frac{gt}{c} \quad (27)$$

Note that, by using

$$\begin{aligned} \cosh \alpha &\sim 1 + \frac{\alpha^2}{2}, \\ \sinh \alpha &\sim \alpha + \frac{\alpha^3}{6}, \end{aligned}$$

it is possible to obtain, in the limit $c \rightarrow \infty$

$$\begin{aligned} X &\rightarrow x + \frac{gt^2}{2}, \\ T &\rightarrow t. \end{aligned}$$

Let us, now, derive the relation between the velocities of a moving particle in I and A . From eqs.(25)-(27) it is possible to obtain

$$dX = dt \cosh \frac{gt}{c} \left[c \left(1 + \frac{gx}{c^2} \right) \tanh \frac{gt}{c} + \frac{dx}{dt} \right], \quad (28)$$

$$dY = dy, \quad dZ = dz, \quad (29)$$

$$dT = dt \cosh \frac{gt}{c} \left[\left(1 + \frac{gx}{c^2} \right) + \frac{\tanh \frac{gt}{c}}{c} \frac{dx}{dt} \right]. \quad (30)$$

Then,

$$\frac{dX}{dT} = \frac{c \left(1 + \frac{gx}{c^2} \right) \tanh \frac{gt}{c} + \frac{dx}{dt}}{\left(1 + \frac{gx}{c^2} \right) + \frac{\tanh \frac{gt}{c}}{c} \frac{dx}{dt}}. \quad (31)$$

In the limit $c \rightarrow \infty$ eq.(31) yields the Newtonian result for the motion in rectilinearly accelerated frame

$$\frac{dX}{dT} \rightarrow gt + \frac{dx}{dt} \equiv v_{\text{abs}} = v_{\text{trasc}} + v_{\text{rel}}. \quad (32)$$

By using eqs.(25)-(27) in

$$(cdT)^2 - (dX)^2 - (dY)^2 - (dZ)^2$$

it is possible to obtain the spacetime interval of the accelerated frame A which is

$$(ds)^2 = (cdt)^2 \left(1 + \frac{gx}{c^2}\right)^2 - (dx)^2 - (dy)^2 - (dz)^2. \quad (33)$$

Let us, now, consider the motion of a particle of mass m in such a space-time. Its equation of motion with respect to the frame A can be obtained, e.g., with the Lagrangian approach from

$$\frac{d\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)}{d\tau} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

and

$$\mathcal{L} = \frac{m}{2} g_{\mu\nu} \dot{x}_\mu \dot{x}_\nu.$$

The overdot means derivation with respect to the proper time τ of the moving particle. For an unidimensional motion along the x axis it can, then, be obtained

$$\frac{d^2 x}{d\tau^2} = -g \left(1 + \frac{gx}{c^2}\right) \left(\frac{dt}{d\tau}\right)^2 \quad (34)$$

where t is the time of the standard clock located at the origin of the accelerated frame A . It is possible to express eq.(34) in terms of t by noting that τ is given, in general, by

$$d\tau = \frac{\sqrt{g_{\mu\nu} dx^\mu dx^\nu}}{c}.$$

Eq.(33) yields

$$d\tau = \sqrt{\left(1 + \frac{gx}{c^2}\right)^2 - \left(\frac{v}{c}\right)^2} dt, \quad (35)$$

where $v^2 = (dx/dt)^2$. Note that the proper time of the moving particle does not depend explicitly on the acceleration; it depends on the position and on

the velocity. Of course, they will come from the solution of the equation of motion which account for the effects of the force F . With eq.(35) eq.(34) becomes

$$\frac{d^2x}{dt^2} = \frac{2g}{c^2 \left(1 + \frac{gx}{c^2}\right)} \left(\frac{dx}{dt}\right)^2 - g \left(1 + \frac{gx}{c^2}\right). \quad (36)$$

It may interesting to note that for $c \rightarrow \infty$ eq.(36) reduces to the Newtonian equation

$$\frac{d^2x}{dt^2} = -g.$$

In order to solve eq.(36), let us pose

$$p = \frac{dx}{dt}; \quad (37)$$

then

$$\frac{d^2x}{dt^2} = p \frac{dp}{dx}.$$

Eq.(36) can, then, be cast into the form

$$\frac{dp}{dx} = \frac{2g}{c^2 \left(1 + \frac{gx}{c^2}\right)} p - \frac{g}{p} \left(1 + \frac{gx}{c^2}\right) \quad (38)$$

whose general solution is

$$p(x) = \pm c \left(1 + \frac{gx}{c^2}\right) \sqrt{1 + C_1 c^6 \left(1 + \frac{gx}{c^2}\right)^2}. \quad (39)$$

By inserting⁴ eq.(39) in eq.(37) allows to obtain

$$\frac{dx}{\left(1 + \frac{gx}{c^2}\right) \sqrt{1 + C_1 c^6 \left(1 + \frac{gx}{c^2}\right)^2}} = c dt,$$

whose general solution is

$$x(t) = -\frac{-c^2 + c^4 C_1 C_2^2 e^{\frac{2g(t-t_0)}{c}} + 2C_2 e^{\frac{g(t-t_0)}{c}}}{g \left[-1 + c^2 C_1 C_2^2 e^{\frac{2g(t-t_0)}{c}}\right]}.$$

The velocity of the particle is

$$\frac{dx(t)}{dt} = \frac{2C_2 e^{\frac{g(t-t_0)}{c}} \left[1 + c^2 C_1 C_2^2 e^{\frac{2g(t-t_0)}{c}}\right]}{c \left[-1 + c^2 C_1 C_2^2 e^{\frac{2g(t-t_0)}{c}}\right]^2}.$$

⁴The + sign must be retained with $dx > 0$ and vice versa because $dt > 0$.

The constants of integration C_1 and C_2 can be determined from the initial conditions $x(t_0) = x_0$, $dx(t_0)/dt = v_0$. It turns out that

$$C_1 = -\frac{1 + \left(\frac{gx_0}{c^2}\right)^2 - \left(\frac{v_0}{c}\right)^2 \left(1 - \frac{2gx_0}{v_0^2 c^2}\right)}{c^6 \left(1 + \frac{gx_0}{c^2}\right)^4},$$

$$C_2 = c^2 \frac{\left(1 + \frac{gx_0}{c^2}\right)^2}{\left(1 + \frac{v_0}{c} + \frac{gx_0}{c^2}\right)}.$$

Then, the position and the velocity of the particle are given by

$$x(t) = \frac{c^2}{g} \left[\frac{\left(1 + \frac{gx_0}{c^2}\right)^2}{\left(1 + \frac{gx_0}{c^2}\right) \cosh \frac{g(t-t_0)}{c} - \frac{v_0}{c} \sinh \frac{g(t-t_0)}{c}} - 1 \right], \quad (40)$$

$$\frac{dx(t)}{dt} = -c \frac{\left(1 + \frac{gx_0}{c^2}\right)^2 \left[\left(1 + \frac{gx_0}{c^2}\right) \tanh \frac{g(t-t_0)}{c} - \frac{v_0}{c} \right]}{\left[\left(1 + \frac{gx_0}{c^2}\right) - \frac{v_0}{c} \tanh \frac{g(t-t_0)}{c} \right]^2}. \quad (41)$$

Note that, for $v_0 = 0$, eqs.(40)-(41) reduces to (8.173) and (8.174) of [10]. Moreover, by expanding the hyperbolic functions to order $\mathcal{O}(g)$, it can also be noted that, for $g \rightarrow 0$,

$$\begin{aligned} x(t) &\rightarrow x_0 + v_0(t - t_0), \\ \frac{dx(t)}{dt} &\rightarrow v_0. \end{aligned}$$

Moreover, for $c \rightarrow \infty$, the Newtonian limit is restored

$$\begin{aligned} x(t) &\rightarrow x_0 + v_0(t - t_0) - \frac{g(t - t_0)^2}{2}, \\ \frac{dx(t)}{dt} &\rightarrow v_0 - g(t - t_0). \end{aligned}$$

Concerning the proper time of the moving particle, eqs.(40)-(41) in eq.(35) yield

$$\tau - \tau_0 = \frac{c}{g} \frac{\sqrt{\left(1 + \frac{gx_0}{c^2}\right)^2 - \left(\frac{v_0}{c}\right)^2} \left(1 + \frac{gx_0}{c^2}\right) \tanh \frac{g(t-t_0)}{c}}{\left[\left(1 + \frac{gx_0}{c^2}\right) - \frac{v_0}{c} \tanh \frac{g(t-t_0)}{c} \right]}. \quad (42)$$

3.2 The motion of the clock (1) with respect to the clock (2)

The results of the previous section will now be used in order to describe the motion of the clock (1) with respect to the clock (2). In this case t is

the proper time $\tau^{(2)}$ of the clock (2), which is now constantly at rest in the origin of A , while τ denotes the proper time $\tau^{(1)}$ of the moving clock (1).

In regard to the proper time of (2), eqs.(25)-(27) yield

$$\tanh \frac{gt}{c} = \frac{gT}{c \left(1 + \frac{gX}{c^2}\right)}. \quad (43)$$

Eq.(43) allows to use the results of Section 2. By using eq.(7) and eq.(8) for all the four steps in which we have subdivided the motion, it is straightforward to obtain

$$\Delta\tau_A^{(2)} = \frac{4c}{g} \operatorname{atanh} \left(\frac{V}{c} \right) = \Delta\tau_I^{(2)}. \quad (44)$$

This result is very important because it shows that *the proper time interval measured by the accelerated clock (2) after its reunion with the clock (1) is the same both in A and in I* , as it can be expected.

The calculation of the proper time of the clock (1) is a bit more involved. To this aim, let us note that in eq.(31) it is always $dX/dT = 0$ for the clock (1), of course; then

$$\frac{dx}{dt} = -c \left(1 + \frac{gx}{c^2}\right) \tanh \frac{gt}{c} \quad (45)$$

must always hold. We will split the motion in four steps.

3.2.1 From $t = t_0$ to $t = t_1$

In this stage the relevant initial conditions are

- $g > 0$
- $x(t_0) = 0$
- $v(t_0) = 0$ (from eq.(31) for $t = 0$ and $dX/dT = 0$)

From eq.(42) one gets

$$\tau_1 - \tau_0 = \frac{c}{g} \tanh \frac{g(t_1 - t_0)}{c}. \quad (46)$$

By using eq.(10) for $t_1 - t_0$ in eq.(46) it is possible to obtain

$$\Delta\tau_A^{(1)} \Big|_{0-1} = \frac{V}{g}. \quad (47)$$

Note that, from eq.(45) for $g > 0$ and $t = t_1$ of eq.(10), it turns out

$$v(t_1) = -V \left(1 + \frac{gl_1}{c^2}\right). \quad (48)$$

3.2.2 From $t = t_1$ to $t = t_2$

In this stage the relevant initial conditions are

- $g < 0$
- $x(t_1) = -l_1$
- $v(t_1) = -V \left(1 + \frac{gl_1}{c^2}\right)$

i.e. at t_1 the same inertial force as before is switched again on but in the opposite direction, so that the clock (1) decelerates until it stops.

The proper time interval of the clock (1) can be obtained from eq.(42) for $x_0 = -l_1$, $v_0 = -V(1 + gl_1/c^2)$, $g = -g$. It is

$$\tau_2 - \tau_1 = \frac{c \left(1 + \frac{gl_1}{c^2}\right) \sqrt{1 - \left(\frac{V}{c}\right)^2} \tanh \frac{g(t_2 - t_1)}{c}}{g \left(1 - \frac{V}{c} \tanh \frac{g(t_2 - t_1)}{c}\right)}. \quad (49)$$

Eq.(15) for $t_2 - t_1$ in eq.(49) yields

$$\Delta\tau_A^{(1)} \Big|_{1-2} = \frac{V \left(1 + \frac{gl_1}{c^2}\right)}{g \sqrt{1 - \left(\frac{V}{c}\right)^2}}. \quad (50)$$

It is necessary to derive an explicit expression for l_1 , as it will become clear later. From eq.(40), evaluated for $x = -l_1$, $x_0 = v_0 = 0$, and eq.(10) for $t_1 - t_0$ it follows

$$l_1 = \frac{c^2}{g} \left[1 - \sqrt{1 - \left(\frac{V}{c}\right)^2} \right]. \quad (51)$$

Finally, note that from eq.(41), evaluated for $t - t_0 = t_2 - t_1$ of eq.(15), $g < 0$ and v_0 of eq.(48), it turns out that $v(t_2) = 0$.

3.2.3 From t_2 to t_3

In this stage the relevant initial conditions are⁵

- $g < 0$

⁵Note that, since (1) and (2) are at rest relative to each other at the inversion point, the maximum distance between the two clocks poses no ambiguity and is the same when measured in I and A ; this is the reason why the same symbol L as before in Section 2 is used here. Moreover, note that from eq.(25) for $t=0$ it follows just $x = X$.

- $x(t_2) = -L$
- $v(t_2) = 0$

i.e. the inertial force continues to act upon the clock (1) along the positive x axis so that it starts accelerating until a velocity related to V is reached at t_3 . The proper time interval of (1) can be obtained from eq.(42) for $x_0 = -L, v_0 = 0, g = -g$. It is

$$\tau_3 - \tau_2 = \frac{c}{g} \left(1 + \frac{gL}{c^2} \right) \tanh \frac{g(t_3 - t_2)}{c}. \quad (52)$$

Eq.(19) for $t_3 - t_2$ and eq.(13) for L in eq.(52) yield

$$\Delta\tau_A^{(1)} \Big|_{2-3} = \left(1 + \frac{gL}{c^2} \right) \frac{V}{g} = \frac{2V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}} - \frac{V}{g}. \quad (53)$$

Note that, from eq.(45) for $g < 0$ and $t = t_3$ of eq.(19), it turns out

$$v(t_3) = V \left(1 - \frac{gl_2}{c^2} \right). \quad (54)$$

3.2.4 From t_3 to t_4

In this stage the relevant initial conditions are

- $g > 0$
- $x(t_3) = -l_2$
- $v(t_3) = V \left(1 - \frac{l_2 g}{c^2} \right)$

i.e. the same force as before is switched again on, but in the opposite direction, at t_3 and the clock (1) is decelerated until it stops at t_4 when it meets the clock (2) again. The proper time interval of the clock (1) can be obtained from eq.(42) for $x_0 = -l_2, v_0 = V(1 - gl_2/c^2), g = g$. It is

$$\tau_4 - \tau_3 = \frac{\frac{c}{g} \left(1 - \frac{gl_2}{c^2} \right) \sqrt{1 - \left(\frac{V}{c}\right)^2} \tanh \frac{g(t_4 - t_3)}{c}}{1 - \frac{V}{c} \tanh \frac{g(t_4 - t_3)}{c}} \quad (55)$$

Eq.(22) for $t_4 - t_3$ in eq.(55) yields

$$\Delta\tau_A^{(1)} \Big|_{3-4} = \frac{V \left(1 - \frac{gl_2}{c^2} \right)}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}}. \quad (56)$$

The explicit expression for l_2 can be obtained from eq.(40), evaluated for $x = 0$, $x_0 = -l_2$, $v_0 = V(1 - gl_2/c^2)$, and eq.(22) for $t_4 - t_3$. Then,

$$l_2 = \frac{c^2}{g} \left[1 - \sqrt{1 - \left(\frac{V}{c}\right)^2} \right] = l_1. \quad (57)$$

3.3 The total proper time interval of the clock (1)

From eq.(47), eq.(50), eq.(53) and eq.(56), and because $l_1 = l_2$ according to eq.(51) and eq.(57), it turns out

$$\Delta\tau_A^{(1)} = \frac{4V}{g\sqrt{1 - \left(\frac{V}{c}\right)^2}} = \Delta\tau_I^{(1)}. \quad (58)$$

This is an important result because it shows that *the proper time interval reckoned by the clock (1) after its reunion with the clock (2) is the same both in I and in A*, as expected.

4 Concluding remarks

In this paper we have studied the clock paradox in the framework of the Special and General Theories of Relativity. We have considered a *rectilinear* (in space) motion of the moving clock during which it is continuously acted upon by a force which, at a certain instant, is reversed, slows it down until inverts its motion, re-accelerates it, inverts once more its action and decelerates again the moving clock until the latter one stops and meets again the rest clock.

In the case of an uniform and constant force *of finite magnitude in the direction of the motion, such force does affect both the proper time of the moving clock and the proper time of the rest clock which sees it moving to-and-fro*.

The expressions for the proper time intervals $\Delta\tau$ measured by a given clock at the clocks' reunion are the *same*, as expected, both in the inertial frame I in which (1) is at rest while (2) performs a special relativistic hyperbolic (in spacetime) motion, and in the accelerated frame A in which (2) is at rest and the General Theory of Relativity has been adopted in order to describe the motion of (1), i.e. $\Delta\tau_I^{(1)} = \Delta\tau_A^{(1)}$ and $\Delta\tau_I^{(2)} = \Delta\tau_A^{(2)}$.

It turns out that

$$\frac{\Delta\tau^{(2)}}{\Delta\tau^{(1)}} = \frac{\sqrt{1 - \beta^2} \operatorname{atanh}\beta}{\beta} < 1,$$

where $\beta \equiv V/c$, i.e., *the moving clock lags always behind the rest clock by an amount which depends only on the speed reached by the moving clock when the force inverts its action*. This result agrees with that found by the authors of [1]: indeed, according to them, the clock with the greater acceleration will mark shorter proper time intervals than the clock with smaller acceleration when they meet again.

In conclusion, in the case of a purely accelerated motion of the clock which moves to-and-fro along a spatial straight line, a differential aging with respect to the rest clock takes place. The moving clock lags always behind the rest clock by an amount which is different with respect to that which occurs when only inertial motion is considered. The Special and General Theories of Relativity are able to explain in a consistent way this feature.

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