We already have a solution for the scalar potential in expression (1.17):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \tag{1.17}$$

To verify directly that this does indeed satisfy the Poisson equation (1.28), we operate with the Laplacian on both sides. Because it turns out that the resulting integrand is singular, we invoke a limiting procedure. Define the "a-potential" $\Phi_a(\mathbf{x})$ by

$$\Phi_a(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{\sqrt{(\mathbf{x} - \mathbf{x}')^2 + a^2}}\, d^3x'$$

The actual potential (1.17) is then the limit of the "a-potential" as $a \to 0$. Taking the Laplacian of the "a-potential" gives

$$\begin{aligned}\nabla^2\Phi_a(\mathbf{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\nabla^2\left(\frac{1}{\sqrt{r^2 + a^2}}\right) d^3x' \\ &= -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')\left[\frac{3a^2}{(r^2 + a^2)^{5/2}}\right] d^3x'\end{aligned} \tag{1.30}$$

where $r = |\mathbf{x} - \mathbf{x}'|$. The square-bracketed expression is the negative Laplacian of $1/\sqrt{r^2 + a^2}$. It is well-behaved everywhere for nonvanishing a, but as a tends to zero it becomes infinite at $r = 0$ and vanishes for $r \neq 0$. It has a volume integral equal to 4π for arbitrary a. For the purposes of integration, divide space into two regions by a sphere of fixed radius R centered on $\mathbf{x}$. Choose R such that $\rho(\mathbf{x}')$ changes little over the interior of the sphere, and imagine a much smaller than R and tending toward zero. If $\rho(\mathbf{x}')$ is such that (1.17) exists, the contribution to the integral (1.30) from the exterior of the sphere will vanish like a^2 as $a \to 0$. We thus need consider only the contribution from inside the sphere. With a Taylor series expansion of the well-behaved $\rho(\mathbf{x}')$ around $\mathbf{x}' = \mathbf{x}$, one finds

$$\nabla^2\Phi_a(\mathbf{x}) = -\frac{1}{\epsilon_0}\int_0^R \frac{3a^2}{(r^2 + a^2)^{5/2}}\left[\rho(\mathbf{x}) + \frac{r^2}{6}\nabla^2\rho + \cdots\right] r^2 dr + O(a^2)$$

Direct integration yields

$$\nabla^2\Phi_a(\mathbf{x}) = -\frac{1}{\epsilon_0}\rho(\mathbf{x})\left(1 + O(a^2/R^2)\right) + O(a^2, a^2\log a)\,\nabla^2\rho + \cdots$$

In the limit $a \to 0$, we obtain the Poisson equation (1.28).

The singular nature of the Laplacian of $1/r$ can be exhibited formally in terms of a Dirac delta function. Since $\nabla^2(1/r) = 0$ for $r \neq 0$ and its volume integral is -4π, we can write the formal equation, $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$ or, more generally,

$$\nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.31}$$

1.8 *Green's Theorem*

If electrostatic problems always involved localized discrete or continuous distributions of charge with no boundary surfaces, the general solution (1.17) would

1.17 is just Coulomb's law

be the most convenient and straightforward solution to any problem. There would be no need of the Poisson or Laplace equation. In actual fact, of course, many, if not most, of the problems of electrostatics involve finite regions of space, with or without charge inside, and with prescribed boundary conditions on the bounding surfaces. These boundary conditions may be simulated by an appropriate distribution of charges outside the region of interest (perhaps at infinity), but (1.17) becomes inconvenient as a means of calculating the potential, except in simple cases (e.g., method of images).

1.17 is just Coulomb's law

To handle the boundary conditions it is necessary to develop some new mathematical tools, namely, the identities or theorems due to George Green (1824). These follow as simple applications of the divergence theorem. The divergence theorem:

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \oint_S \mathbf{A} \cdot \mathbf{n}\, da$$

applies to any well-behaved vector field $\mathbf{A}$ defined in the volume V bounded by the closed surface S. Let $\mathbf{A} = \phi\, \nabla\psi$, where ϕ and ψ are arbitrary scalar fields. Now

$$\nabla \cdot (\phi\, \nabla\psi) = \phi\, \nabla^2\psi + \nabla\phi \cdot \nabla\psi \tag{1.32}$$

and

$$\phi\, \nabla\psi \cdot \mathbf{n} = \phi \frac{\partial\psi}{\partial n} \tag{1.33}$$

defined

where $\partial/\partial n$ is the normal derivative at the surface S (directed outward from inside the volume V). When (1.32) and (1.33) are substituted into the divergence theorem, there results *Green's first identity*:

$$\int_V (\phi\, \nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \oint_S \phi \frac{\partial\psi}{\partial n}\, da \tag{1.34}$$

If we write down (1.34) again with ϕ and ψ interchanged, and then subtract it from (1.34), the $\nabla\phi \cdot \nabla\psi$ terms cancel, and we obtain Green's second identity or *Green's theorem*:

$$\int_V (\phi\, \nabla^2\psi - \psi\, \nabla^2\phi)\, d^3x = \oint_S \left[\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right] da \tag{1.35}$$

this choice of 1/R is one of many possible functions, all called Greens fcns and all of which must satisfy eqn 1.39 to 1.41

The Poisson differential equation for the potential can be converted into an integral equation if we choose a particular ψ, namely $1/R \equiv 1/|\mathbf{x} - \mathbf{x}'|$, where $\mathbf{x}$ is the observation point and $\mathbf{x}'$ is the integration variable. Further, we put $\phi = \Phi$, the scalar potential, and make use of $\nabla^2\Phi = -\rho/\epsilon_0$. From (1.31) we know that $\nabla^2(1/R) = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, so that (1.35) becomes

$$\int_V \left[-4\pi\Phi(\mathbf{x}')\, \delta(\mathbf{x} - \mathbf{x}') + \frac{1}{\epsilon_0 R}\rho(\mathbf{x}')\right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right] da'$$

R = | x - x' |

If the point $\mathbf{x}$ lies within the volume V, we obtain:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R}\, d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da' \tag{1.36}$$

the benefit of this is it can handle point charges AND BOUNDARY CONDITIONS, which coulomb's law can't do

If $\mathbf{x}$ lies outside the surface S, the left-hand side of (1.36) is zero.* [Note that this is consistent with the interpretation of the ~~surface integral as being the potential due to a surface-charge density~~ $\sigma = \epsilon_0\, \partial\Phi/\partial n'$ and a dipole layer $D = -\epsilon_0 \Phi$. The discontinuities in electric field and potential (1.22) and (1.27) across the surface then lead to zero field and zero potential outside the volume V.]

Two remarks are in order about result (1.36). First, if the surface S goes to infinity and the electric field on S falls off faster than R^{-1}, then the surface integral vanishes and (1.36) reduces to the familiar result (1.17). Second, for a charge-free volume, the potential anywhere inside the volume (a solution of the Laplace equation) is expressed in (1.36) in terms of the potential and its normal derivative only on the surface of the volume. This rather surprising result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both Φ and $\partial\Phi/\partial n$ (*Cauchy boundary conditions*) is an overspecification of the problem. This is discussed in detail in the next sections, where techniques yielding solutions for appropriate boundary conditions are developed using Green's theorem (1.35).

In the next 3 chapters Jackson makes it clear that one or all of the "enclosing" surfaces can be located at infinity… allowing a pseudo "open" geometry.

1.9 *Uniqueness of the Solution with Dirichlet or Neumann Boundary Conditions*

What boundary conditions are appropriate for the Poisson (or Laplace) equation to ensure that a unique and well-behaved (i.e., physically reasonable) solution will exist inside the bounded region? Physical experience leads us to believe that specification of the potential on a closed surface (e.g., a system of conductors held at different potentials) defines a unique potential problem. This is called a *Dirichlet problem*, or *Dirichlet boundary conditions*. Similarly it is plausible that specification of the electric field (normal derivative of the potential) everywhere on the surface (corresponding to a given surface-charge density) also defines a unique problem. Specification of the normal derivative is known as the *Neumann boundary condition*. We now proceed to prove these expectations by means of Green's first identity (1.34).

We want to show the uniqueness of the solution of the Poisson equation, $\nabla^2\Phi = -\rho/\epsilon_0$, inside a volume V subject to either Dirichlet or Neumann boundary conditions on the closed bounding surface S. We suppose, to the contrary, that there exist two solutions Φ_1 and Φ_2 satisfying the same boundary conditions. Let

$$U = \Phi_2 - \Phi_1 \tag{1.37}$$

Then $\nabla^2 U = 0$ inside V, and $U = 0$ or $\partial U/\partial n = 0$ on S for Dirichlet and Neumann boundary conditions, respectively. From Green's first identity (1.34), with $\phi = \psi = U$, we find

$$\int_V (U\, \nabla^2 U + \nabla U \cdot \nabla U)\, d^3x = \oint_S U \frac{\partial U}{\partial n}\, da \tag{1.38}$$

*The reader may complain that (1.36) has been obtained in an illegal fashion since $1/|\mathbf{x} - \mathbf{x}'|$ is not well-behaved inside the volume V. Rigor can be restored by using a limiting process, as in the preceding section, or by excluding a small sphere around the offending point, $\mathbf{x} = \mathbf{x}'$. The result is still (1.36).

With the specified properties of U, this reduces (for both types of boundary condition) to:

$$\int_V |\nabla U|^2 \, d^3x = 0$$

which implies $\nabla U = 0$. Consequently, inside V, U is constant. For Dirichlet boundary conditions, $U = 0$ on S so that, inside V, $\Phi_1 = \Phi_2$ and the solution is unique. Similarly, for Neumann boundary conditions, the solution is unique, apart from an unimportant arbitrary additive constant.

From the right-hand side of (1.38) it is evident that there is also a unique solution to a problem with mixed boundary conditions (i.e., Dirichlet over part of the surface S, and Neumann over the remaining part).

It should be clear that a solution to the Poisson equation with both Φ and $\partial\Phi/\partial n$ specified arbitrarily on a closed boundary (Cauchy boundary conditions) does not exist, since there are unique solutions for Dirichlet and Neumann conditions separately and these will in general not be consistent. This can be verified with (1.36). With arbitrary values of Φ and $\partial\Phi/\partial n$ inserted on the right-hand side, it can be shown that the values of $\Phi(\mathbf{x})$ and $\nabla\Phi(\mathbf{x})$ as $\mathbf{x}$ approaches the surface are in general inconsistent with the assumed boundary values. The question of whether Cauchy boundary conditions on an *open* surface define a unique electrostatic problem requires more discussion than is warranted here. The reader may refer to *Morse and Feshbach* (Section 6.2, pp. 692–706) or to *Sommerfeld* (*Partial Differential Equations in Physics*, Chapter II) for a detailed discussion of these questions. The conclusion is that electrostatic problems are specified only by Dirichlet *or* Neumann boundary conditions on a closed surface (part or all of which may be at infinity, of course).

1.10 Formal Solution of Electrostatic Boundary-Value Problem with Green Function

The solution of the Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S can be obtained by means of Green's theorem (1.35) and so-called Green functions.

In obtaining result (1.36)—not a solution—we chose the function ψ to be $1/|\mathbf{x} - \mathbf{x}'|$, it being the potential of a unit point source, satisfying the equation:

the 4 pi comes from the definition of Dirac Delta Function needing volume integration, and if you volume integrate the left-hand side, you get a surface integral via the divergence theorem which produces 4 pi r^2

secion 1.8 called this 1/R

$$\nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.31}$$

The function $1/|\mathbf{x} - \mathbf{x}'|$ is only one of a class of functions depending on the variables $\mathbf{x}$ and $\mathbf{x}'$, and called *Green functions*, which satisfy (1.31). In general,

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{1.39}$$

we need to satisfy 1.31 in order to cull PHI out of the integral in Greens 2nd Identity

where

All the example Green functions I've seen in this book include a 1/R term to satisfy eqn 1.39 even though the 1/R term may be disguised as a sum of other functions that are equivalent to 1/R. I guess that totally different functions (not 1/R) might be able to satisfy eqn 1.39 but I haven't seen any yet.

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \tag{1.40}$$

with the function F satisfying the Laplace equation inside the volume V:

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0 \tag{1.41}$$

ok, so F(x,x') has more freedom than G(x,x') because it doesn't need to be a Dirac Delta function at x = x'

In facing the problem of satisfying the prescribed boundary conditions on Φ *or* $\partial\Phi/\partial n$, we can find the key by considering result (1.36). As has been pointed out already, this is not a solution satisfying the correct type of boundary conditions because both Φ *and* $\partial\Phi/\partial n$ appear in the surface integral. It is at best an integral relation for Φ. With the generalized concept of a Green function and its additional freedom [via the function $F(\mathbf{x}, \mathbf{x}')$], there arises the possibility that we can use Green's theorem with $\psi = G(\mathbf{x}, \mathbf{x}')$ and choose $F(\mathbf{x}, \mathbf{x}')$ to eliminate one or the other of the two surface integrals, obtaining a result that involves only Dirichlet or Neumann boundary conditions. Of course, if the necessary $G(\mathbf{x}, \mathbf{x}')$ depended in detail on the exact form of the boundary conditions, the method would have little generality. As will be seen immediately, this is not required, and $G(\mathbf{x}, \mathbf{x}')$ satisfies rather simple boundary conditions on S.

With Green's theorem (1.35), $\phi = \Phi$, $\psi = G(\mathbf{x}, \mathbf{x}')$, and the specified properties of G (1.39), it is simple to obtain the generalization of (1.36):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x}, \mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}\right] da' \tag{1.42}$$

The freedom available in the definition of G (1.40) means that we can make the surface integral depend only on the chosen type of boundary conditions. Thus, for *Dirichlet boundary conditions* we demand:

this type of B.C. specifies a definite value for the solution on the boundary.

$$G_D(\mathbf{x}, \mathbf{x}') = 0 \qquad \text{for } \mathbf{x}' \text{ on } S \tag{1.43}$$

Then the first term in the surface integral in (1.42) vanishes and the solution is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x}, \mathbf{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D}{\partial n'}\, da' \tag{1.44}$$

this type of B.C. specifies a definite value of the derivative of the solution on the boundary.

For *Neumann boundary conditions* we must be more careful. The obvious choice of boundary condition on $G(\mathbf{x}, \mathbf{x}')$ seems to be

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = 0 \qquad \text{for } \mathbf{x}' \text{ on } S$$

i.e. Divergence Theorem

since that makes the second term in the surface integral in (1.42) vanish, as desired. But an application of Gauss's theorem to (1.39) shows that

the Divergence Theorem means this integral is equal to a volume integral of the div grad (G), which by 1.39 equals 4 pi Dirac (x - x') and if we assume that x = x' occurs in the volume then ...

$$\oint_S \frac{\partial G}{\partial n'}\, da' = -4\pi$$

Consequently the simplest allowable boundary condition on G_N is

$$\frac{\partial G_N}{\partial n'}(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{S} \qquad \text{for } \mathbf{x}' \text{ on } S \tag{1.45}$$

where S is the total area of the boundary surface. Then the solution is

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_N(\mathbf{x}, \mathbf{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}G_N\, da' \tag{1.46}$$

where $\langle\Phi\rangle_S$ is the average value of the potential over the whole surface. The customary Neumann problem is the so-called exterior problem in which the vol-

Confusing, but the author suggesting a volume created by the annulus space between concentric spheres, like layers of an onion. In other words, the surface doesn't need to be continuous/contiguous — only requirement is that it bounds a volume

ume V is bounded by two surfaces, one closed and finite, the other at infinity. Then the surface area S is infinite; the boundary condition (1.45) becomes homogeneous; the average value $\langle \Phi \rangle_S$ vanishes.

because Phi at infinity is zero

We note that the Green functions satisfy simple boundary conditions (1.43) or (1.45) which do not depend on the detailed form of the Dirichlet (or Neumann) boundary values. Even so, it is often rather involved (if not impossible) to determine $G(\mathbf{x}, \mathbf{x}')$ because of its dependence on the shape of the surface S. We will encounter such problems in Chapters 2 and 3.

The mathematical symmetry property $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$ can be proved for the Green functions satisfying the Dirichlet boundary condition (1.43) by means of Green's theorem with $\phi = G(\mathbf{x}, \mathbf{y})$ and $\psi = G(\mathbf{x}', \mathbf{y})$, where $\mathbf{y}$ is the integration variable. Since the Green function, as a function of one of its variables, is a potential due to a unit point source, the symmetry merely represents the physical interchangeability of the source and the observation points. For Neumann boundary conditions the symmetry is not automatic, but can be imposed as a separate requirement.*

As a final, important remark we note the physical meaning of $F(\mathbf{x}, \mathbf{x}')/4\pi\epsilon_0$. It is a solution of the Laplace equation inside V and so represents the potential of a system of charges *external to the volume V*. It can be thought of as the potential due to an external distribution of charges chosen to satisfy the homogeneous boundary conditions of zero potential (or zero normal derivative) on the surface S when combined with the potential of a point charge at the source point $\mathbf{x}'$. Since the potential at a point $\mathbf{x}$ on the surface due to the point charge depends on the position of the source point, the external distribution of charge $F(\mathbf{x}, \mathbf{x}')$ must also depend on the "parameter" $\mathbf{x}'$. From this point of view, we see that the method of images (to be discussed in Chapter 2) is a physical equivalent of the determination of the appropriate $F(\mathbf{x}, \mathbf{x}')$ to satisfy the boundary conditions (1.43) or (1.45). For the Dirichlet problem with conductors, $F(\mathbf{x}, \mathbf{x}')/4\pi\epsilon_0$ can also be interpreted as the potential due to the surface-charge distribution induced on the conductors by the presence of a point charge at the source point $\mathbf{x}'$.

1.11 Electrostatic Potential Energy and Energy Density; Capacitance

In Section 1.5 it was shown that the product of the scalar potential and the charge of a point object could be interpreted as potential energy. More precisely, if a point charge q_i is brought from infinity to a point $\mathbf{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the work done on the charge (and hence its potential energy) is given by

$$W_i = q_i \Phi(\mathbf{x}_i) \tag{1.47}$$

The potential Φ can be viewed as produced by an array of $(n - 1)$ charges $q_j (j = 1, 2, \ldots, n - 1)$ at positions $\mathbf{x}_j$. Then

$$\Phi(\mathbf{x}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{1.48}$$

*See K.-J. Kim and J. D. Jackson, *Am. J. Phys.* **61**, (12) 1144–1146 (1993).