

Lecture Notes on Kinematics

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For details see:

- 1) Ya.V. Tatarinov: Lectures on Classical Dynamics. MSU Press, 1984(in Russian);
- 2) S.V. Bolotin, A.V. Karapetyan, E.I. Kugushev, D.V. Treschev: Theoretical Mechanics. Moscow, "Academia", 2010(in Russian).

1. Velocity and Acceleration of a Point

Let us fix a positively oriented Cartesian coordinate system $Oxyz$ in the physical $\mathbb{R}^3$.

The law of motion of a point M with radius vector $\mathbf{r}_M(t) = \mathbf{OM}(t)$ is defined as the mapping

$$t \mapsto \mathbf{r}_M(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z, \quad x(\cdot), y(\cdot), z(\cdot) \in C^2[0, T].$$

Here $\mathbf{e}_x$ and so on denote the unit basis vectors of the corresponding axes.

The trajectory of a point M is the set of points

$$\{\mathbf{r}_M(t) \mid t \in [0, T]\} \subset \mathbb{R}^3,$$

which is typically a piecewise smooth curve.

The velocity and acceleration of the point M relative to $Oxyz$ are defined respectively as the vectors

$$\mathbf{v}_M = \frac{d\mathbf{r}_M}{dt} = \dot{\mathbf{r}}_M = \dot{x}(t)\mathbf{e}_x + \dot{y}(t)\mathbf{e}_y + \dot{z}(t)\mathbf{e}_z, \quad \mathbf{a}_M = \dot{\mathbf{v}}_M.$$

In the following, unless specified otherwise, velocities and accelerations are considered relative to the system $Oxyz$.

2. Kinematics of a Rigid Body

DEFINITION 1. Let $D \subset \mathbb{R}^3$ be a non-empty open set and consider a pair of functions

$$\mathbf{r}_A \in C^2([0, T], \mathbb{R}^3), \quad U \in C^2([0, T], \text{SO}(3)), \quad U(0) = E, \quad \mathbf{r}_A(0) \in \overline{D}.$$

By definition, a rigid body consists of points B moving according to the law

$$\mathbf{r}_B(t) = \mathbf{r}_A(t) + U(t)(\mathbf{r}_B(0) - \mathbf{r}_A(0)), \quad t \in [0, T], \quad (1)$$

where the initial condition $\mathbf{r}_B(0) \in \overline{D}$.

The law of motion of point B will not change if, instead of the radius vector $\mathbf{r}_A$ in formula (1), we take the radius vector of any other point belonging to the rigid body.

DEFINITION 2. A vector $\mathbf{u}(t)$ is said to be attached to (or embedded in) the rigid body if there exist two points of the rigid body P, N and a number λ such that $\mathbf{u}(t) = \lambda(\mathbf{r}_P(t) - \mathbf{r}_N(t))$, $t \in [0, T]$.

THEOREM 1 (Euler's Formula). *There exists a unique (axial) vector $\boldsymbol{\omega}$ such that for any two points A, B of the rigid body, the following equality holds*

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{AB}, \quad \mathbf{AB} = \mathbf{r}_B - \mathbf{r}_A.$$

DEFINITION 3. *The vector $\boldsymbol{\omega}$ is called the angular velocity of the rigid body. The vector $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ is called the angular acceleration of the rigid body.*

Proof of Theorem 1. Differentiating equality (1), we find

$$\mathbf{v}_B(t) = \mathbf{v}_A(t) + \dot{U}(t)(\mathbf{r}_B(0) - \mathbf{r}_A(0)) = \mathbf{v}_A(t) + \dot{U}(t)U^{-1}(t)(\mathbf{r}_B(t) - \mathbf{r}_A(t)).$$

The operator $\Omega = \dot{U}U^{-1} = \dot{U}U^*$ is skew-symmetric:

$$UU^* = E, \quad \frac{d}{dt}(UU^*) = \dot{U}U^* + U\dot{U}^* = 0 \implies \Omega^* = -\Omega.$$

From linear algebra [1], it is known that a skew-symmetric operator in $\mathbb{R}^3$ corresponds to a unique axial vector $\boldsymbol{\omega}$ such that

$$\Omega \mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

Let us show that the angular velocity vector is uniquely defined. Assume the contrary: there exists another vector $\boldsymbol{\omega}'$ satisfying Euler's formula

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}' \times \mathbf{AB}$$

for any two points A, B of the rigid body. Subtracting one formula from the other gives:

$$(\boldsymbol{\omega}' - \boldsymbol{\omega}) \times \mathbf{AB} = 0.$$

It remains to choose a non-zero vector $\mathbf{AB}$ perpendicular to $\boldsymbol{\omega}' - \boldsymbol{\omega}$. Contradiction.

The theorem is proved.

COROLLARY 1. *Suppose that at some moment in time $\mathbf{v}_A = 0$ with $\boldsymbol{\omega} \neq 0$. Then the set of points of the rigid body whose instantaneous velocities are equal to zero forms a straight line passing through point A parallel to the vector $\boldsymbol{\omega}$. This line is called the instantaneous axis of rotation. The velocities of all other points of the body at this moment are perpendicular to this axis.*

The motion of a rigid body is called translational if the velocities of all its points are equal. The motion of a rigid body is translational if and only if $\boldsymbol{\omega} = 0$.

Let us attach a positively oriented orthonormal frame $\mathbf{e}_1(t)\mathbf{e}_2(t)\mathbf{e}_3(t)$ to the rigid body.

LEMMA 1. *The following formula holds¹*

$$\boldsymbol{\omega} = (\dot{\mathbf{e}}_2, \mathbf{e}_3)\mathbf{e}_1 + (\dot{\mathbf{e}}_3, \mathbf{e}_1)\mathbf{e}_2 + (\dot{\mathbf{e}}_1, \mathbf{e}_2)\mathbf{e}_3. \quad (2)$$

Indeed, let us rewrite Euler's formula as follows

$$\frac{d}{dt}\mathbf{AB} = \boldsymbol{\omega} \times \mathbf{AB}. \quad (3)$$

In particular, *Poisson's formulas* hold:

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \quad i = 1, 2, 3. \quad (4)$$

Now, to verify the statement of the lemma, it is sufficient to substitute (4) into the right-hand side of (2) and use the properties of the triple product.

The lemma is proved.

¹Note that formula (2) as such is not suitable as a definition of angular velocity, since its right-hand side a priori depends on the choice of the frame attached to the rigid body.

THEOREM 2 (Rivals' Formula). *The following formula holds*

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\varepsilon} \times \mathbf{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{AB}).$$

Indeed, this formula is obtained by direct differentiation of Euler's formula using (3).

3. Rotation of a Rigid Body Around a Fixed Axis

Consider an important special case. Suppose that a rigid body rotates around a fixed axis Oz by an angle $\varphi = \varphi(t)$ such that for some positively oriented frame $\{\mathbf{e}_i\}$ attached to the rigid body, the following formulas hold: $\mathbf{e}_3 \equiv \mathbf{e}_z$,

$$\mathbf{e}_1 = \cos \varphi \mathbf{e}_x + \sin \varphi \mathbf{e}_y, \quad \mathbf{e}_2 = \cos \varphi \mathbf{e}_y - \sin \varphi \mathbf{e}_x.$$

Differentiating the first formula:

$$\dot{\mathbf{e}}_1 = \dot{\varphi}(-\sin \varphi \mathbf{e}_x + \cos \varphi \mathbf{e}_y) = \dot{\varphi} \mathbf{e}_2.$$

Similarly: $\dot{\mathbf{e}}_2 = -\dot{\varphi} \mathbf{e}_1$, $\dot{\mathbf{e}}_3 = 0$.

Substituting these formulas into (2), we find

$$\boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_3 = \dot{\varphi} \mathbf{e}_z. \quad (5)$$

4. Kinematics of Relative Motion

Suppose that a positively oriented Cartesian coordinate system $Q\xi\eta\zeta$ moves relative to $Oxyz$ with angular velocity $\boldsymbol{\omega}^e(t)$ and such that $\mathbf{OQ} = \mathbf{OQ}(t)$. Note that a Cartesian coordinate system can be considered as attached to some rigid body.

Let us introduce the notation

$$\mathbf{e}_1 = \mathbf{e}_\xi, \quad \mathbf{e}_2 = \mathbf{e}_\eta, \quad \mathbf{e}_3 = \mathbf{e}_\zeta.$$

Let $\mathbf{f}(t)$ be some time-dependent vector. We can write

$$\mathbf{f} = f_x(t)\mathbf{e}_x + f_y(t)\mathbf{e}_y + f_z(t)\mathbf{e}_z = f^i(t)\mathbf{e}_i. \quad (6)$$

The derivative of this vector with respect to the system $Oxyz$ will be considered, as above, to be the vector

$$\dot{\mathbf{f}} = \dot{f}_x \mathbf{e}_x + \dot{f}_y \mathbf{e}_y + \dot{f}_z \mathbf{e}_z;$$

and the derivative with respect to the system $Q\xi\eta\zeta$ to be the vector

$$\frac{\delta \mathbf{f}}{\delta t} := \dot{f}^i \mathbf{e}_i.$$

LEMMA 2. *The following formula holds*

$$\dot{\mathbf{f}} = \frac{\delta \mathbf{f}}{\delta t} + \boldsymbol{\omega}^e \times \mathbf{f}. \quad (7)$$

Indeed, this formula is obtained by direct differentiation of equality (6) using Poisson's formulas. Let us investigate the relative motion of a point M .

DEFINITION 4. *The relative velocity of point M is defined as its velocity relative to the system $Q\xi\eta\zeta$:*

$$\mathbf{v}^r = \frac{\delta \mathbf{QM}}{\delta t}.$$

The transport (or frame) velocity of point M is defined as the velocity of the point of the system $Q\xi\eta\zeta$ that coincides with M at the given moment of time. It is denoted by $\mathbf{v}^e$.

The relative acceleration of point M is defined as its acceleration relative to the system $Q\xi\eta\zeta$:

$$\mathbf{a}^r = \frac{\delta \mathbf{v}^r}{\delta t}.$$

The transport (or frame) acceleration of point M is defined as the acceleration of the point of the system $Q\xi\eta\zeta$ that coincides with M at the given moment of time. It is denoted by $\mathbf{a}^e$.

The Coriolis acceleration of point M is defined as the vector $\mathbf{a}^c = 2(\boldsymbol{\omega}^e \times \mathbf{v}^r)$.

Note that, generally speaking, $\mathbf{a}^e \neq \dot{\mathbf{v}}^e$.

THEOREM 3. *The following formulas hold*

$$\mathbf{v}_M = \mathbf{v}^r + \mathbf{v}^e, \quad (8)$$

$$\mathbf{a}_M = \mathbf{a}^r + \mathbf{a}^e + \mathbf{a}^c. \quad (9)$$

Proof. Let us differentiate the identity

$$\mathbf{OM} = \mathbf{OQ} + \mathbf{QM},$$

using (7):

$$\mathbf{v}_M = \mathbf{v}_Q + \frac{\delta \mathbf{QM}}{\delta t} + \boldsymbol{\omega}^e \times \mathbf{QM} = \mathbf{v}_Q + \mathbf{v}^r + \boldsymbol{\omega}^e \times \mathbf{QM}. \quad (10)$$

By Euler's formula

$$\mathbf{v}_Q + \boldsymbol{\omega}^e \times \mathbf{QM} = \mathbf{v}^e.$$

Formula (8) is proved.

Let us verify formula (9). Differentiating (10):

$$\mathbf{a}_M = \mathbf{a}_Q + \frac{\delta \mathbf{v}^r}{\delta t} + \boldsymbol{\omega}^e \times \mathbf{v}^r + \boldsymbol{\varepsilon}^e \times \mathbf{QM} + \boldsymbol{\omega}^e \times \left(\frac{\delta \mathbf{QM}}{\delta t} + \boldsymbol{\omega}^e \times \mathbf{QM} \right), \quad \boldsymbol{\varepsilon}^e = \dot{\boldsymbol{\omega}}^e.$$

By Rivals' formula

$$\mathbf{a}_Q + \boldsymbol{\varepsilon}^e \times \mathbf{QM} + \boldsymbol{\omega}^e \times (\boldsymbol{\omega}^e \times \mathbf{QM}) = \mathbf{a}^e.$$

The theorem is proved.

Suppose that a rigid body moves relative to the system $Q\xi\eta\zeta$ with angular velocity $\boldsymbol{\omega}^r$, and relative to the system $Oxyz$ with angular velocity $\boldsymbol{\omega}$.

THEOREM 4. *The following formula holds $\boldsymbol{\omega} = \boldsymbol{\omega}^e + \boldsymbol{\omega}^r$.*

Proof. Take arbitrary points A, B of the rigid body. Using the velocity addition formula (8):

$$\mathbf{v}_B = \mathbf{v}_B^r + \mathbf{v}_B^e.$$

Applying Euler's formula to the rigid body, we find:

$$\mathbf{v}_B^r = \mathbf{v}_A^r + \boldsymbol{\omega}^r \times \mathbf{AB}.$$

Applying Euler's formula to the system $Q\xi\eta\zeta$, we find:

$$\mathbf{v}_B^e = \mathbf{v}_A^e + \boldsymbol{\omega}^e \times \mathbf{AB}.$$

Putting it all together, we have:

$$\mathbf{v}_B = \mathbf{v}_A + (\boldsymbol{\omega}^r + \boldsymbol{\omega}^e) \times \mathbf{AB}.$$

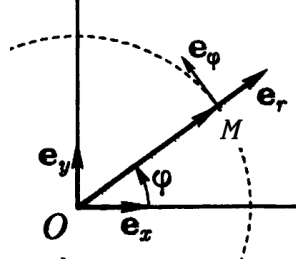
By the uniqueness of the angular velocity, we obtain the required result.

The theorem is proved.

5. Velocity and Acceleration of a Point in Polar Coordinates

Let a point M move in the Oxy plane, where polar coordinates are introduced:

$$x = r \cos \varphi, \quad y = r \sin \varphi.$$



By formula (5), the angular velocity of the frame $\mathbf{e}_r \mathbf{e}_\varphi \mathbf{e}_z$ is equal to $\boldsymbol{\omega} = \dot{\varphi} \mathbf{e}_z$. From which, by Poisson's formulas:

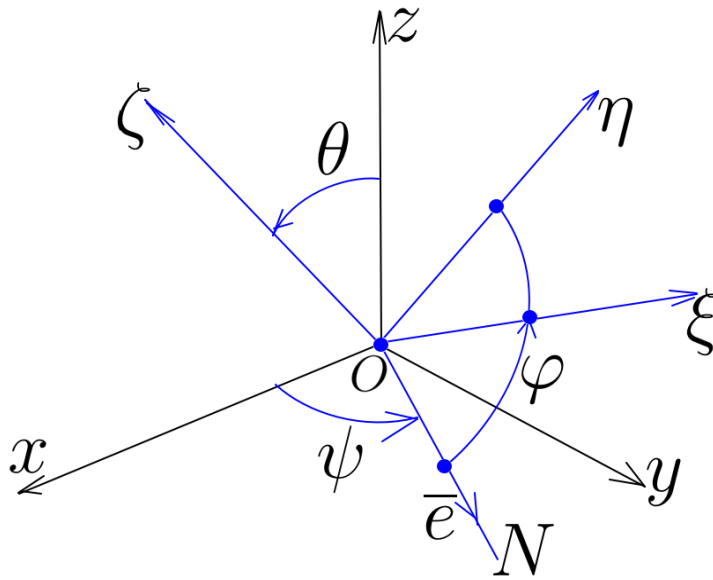
$$\dot{\mathbf{e}}_r = \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\varphi} \mathbf{e}_\varphi, \quad \dot{\mathbf{e}}_\varphi = \boldsymbol{\omega} \times \mathbf{e}_\varphi = -\dot{\varphi} \mathbf{e}_r.$$

Now the velocity and acceleration of point M in polar coordinates are calculated by differentiating the radius vector $\mathbf{OM} = r \mathbf{e}_r$:

$$\mathbf{v}_M = \frac{d}{dt} \mathbf{OM} = \dot{r} \mathbf{e}_r + r \dot{\varphi} \mathbf{e}_\varphi, \quad \mathbf{a}_M = \frac{d}{dt} \mathbf{v}_M = (\ddot{r} - r \dot{\varphi}^2) \mathbf{e}_r + (r \ddot{\varphi} + 2\dot{r} \dot{\varphi}) \mathbf{e}_\varphi.$$

6. Euler Angles and Euler's Kinematic Formulas

The Euler angles determine the position of a rigid body with a fixed point O relative to the fixed frame $Oxyz$. Let us attach a positively oriented orthonormal frame $O\xi\eta\zeta$ to the rigid body.



The line ON (the line of nodes) lies in the intersection of the planes $O\xi\eta$ and Oxy . It is assumed that these planes do not coincide. The Euler angles

$$\theta \in (0, \pi), \quad \varphi \pmod{2\pi}, \quad \psi \pmod{2\pi}$$

are shown in the figure.

Let I denote the coordinate system associated with the axes ON and Oz ;

Let II denote the coordinate system associated with the axes ON and $O\zeta$;

By formula (5), the angular velocity of I relative to $Oxyz$ is

$$\boldsymbol{\omega}_1 = \dot{\psi} \mathbf{e}_z;$$

the angular velocity of II relative to I is

$$\boldsymbol{\omega}_2 = \dot{\theta} \mathbf{e}, \quad \mathbf{e} = \cos \varphi \mathbf{e}_\xi - \sin \varphi \mathbf{e}_\eta;$$

the angular velocity of $O\xi\eta\zeta$ relative to II is

$$\boldsymbol{\omega}_3 = \dot{\varphi} \mathbf{e}_\zeta.$$

By Theorem 4, the angular velocity of the frame $O\xi\eta\zeta$ relative to $Oxyz$ is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3.$$

Note that the plane $O\zeta z$ is perpendicular to the line ON , therefore

$$\mathbf{e}_z = \cos \theta \mathbf{e}_\zeta + \sin \theta (\mathbf{e}_\zeta \times \mathbf{e}).$$

Thus,

$$\begin{aligned} \boldsymbol{\omega} &= (\dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi) \mathbf{e}_\xi \\ &\quad + (\dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi) \mathbf{e}_\eta \\ &\quad + (\dot{\psi} \cos \theta + \dot{\varphi}) \mathbf{e}_\zeta. \end{aligned}$$

7. Appendix

7.1. A Simply Useful Fact. The projections of the velocities of two points A and B of a rigid body onto the straight line connecting them are equal:

$$(\mathbf{v}_A, \mathbf{e}) = (\mathbf{v}_B, \mathbf{e}), \quad \mathbf{e} = \frac{\mathbf{AB}}{|\mathbf{AB}|}.$$

7.2. Kinematics of the Frenet Frame. Let a curve γ be given parametrically by the radius vector $\mathbf{r} = \mathbf{r}(s)$, where s is the arc length parameter:

$$\frac{d}{ds} \mathbf{r} = \mathbf{r}', \quad |\mathbf{r}'| \equiv 1.$$

Suppose that the vectors $\boldsymbol{\tau} = \mathbf{r}'$ and $\mathbf{r}''$ are linearly independent. Note that $\mathbf{r}' \perp \mathbf{r}''$.

Define the curvature of the curve by the formula $k(s) = |\mathbf{r}''|$.

The Frenet formulas:

$$\boldsymbol{\tau}'(s) = k(s) \mathbf{n}(s), \quad \mathbf{n}'(s) = -k(s) \boldsymbol{\tau}(s) - \kappa(s) \mathbf{b}(s), \quad \mathbf{b}'(s) = \kappa(s) \mathbf{n}(s).$$

Here $\mathbf{b} = \boldsymbol{\tau} \times \mathbf{n}$.

Let us decompose the radius vector with respect to the Frenet frame: $\mathbf{r} = x\boldsymbol{\tau} + y\mathbf{n} + z\mathbf{b}$.

Differentiating this formula and substituting the result into the equality $\mathbf{r}' = \boldsymbol{\tau}$, we obtain useful relations:

$$x' = ky + 1, \quad y' = -kx - \kappa z, \quad z' = \kappa y.$$

Suppose that the Frenet frame $M\boldsymbol{\tau}\mathbf{n}\mathbf{b}$ is constructed at point M , which moves along the curve according to the law $\mathbf{r}_M = \mathbf{r}(s(t))$. Then the angular velocity of the Frenet frame is calculated by formula (2):

$$\boldsymbol{\omega} = \dot{s}(-\kappa\boldsymbol{\tau} + k\mathbf{b}), \quad \left(\frac{d}{dt} = \dot{s}\frac{d}{ds}\right).$$

In particular, if the curve is planar ($\kappa = 0$), then from formula (5) we find: $k(s) = \varphi'$.

The velocity and acceleration of point M in such motion are given by the formulas

$$\mathbf{v}_M = \dot{\mathbf{r}}_M = \mathbf{r}'\dot{s} = \dot{s}\boldsymbol{\tau}, \quad \mathbf{a}_M = \ddot{s}\boldsymbol{\tau} + \dot{s}^2k\mathbf{n}.$$

7.3. Rolling Without Slipping. Let a convex rigid body roll without slipping with angular velocity $\boldsymbol{\omega}$ on a fixed plane.

Let P denote the point of the body that touches the plane at the given moment of time. Rolling without slipping means by definition that $\mathbf{v}_P = 0$.

Let M denote the geometric point of contact between the body and the plane. The point of contact traces a curve on the plane during the motion.

THEOREM 5. *The following formula holds $\mathbf{a}_P = -\boldsymbol{\omega} \times \mathbf{v}_M$.*

Proof. Let us note the following formula

$$\frac{d\mathbf{v}^e}{dt} = \mathbf{a}^e + \boldsymbol{\omega}^e \times \mathbf{v}^r. \quad (11)$$

It is obtained by differentiating formula (8):

$$\mathbf{a}_M = \frac{\delta\mathbf{v}^r}{\delta t} + \boldsymbol{\omega}^e \times \mathbf{v}^r + \frac{d\mathbf{v}^e}{dt} = \mathbf{a}^r + \boldsymbol{\omega}^e \times \mathbf{v}^r + \frac{d\mathbf{v}^e}{dt}$$

and comparing this result with (9).

Let us attach the system $Q\xi\eta\zeta$ to the body. Then

$$\mathbf{v}_M = \mathbf{v}_M^r + \mathbf{v}_M^e = \mathbf{v}_M^r.$$

Here we used the definition of the transport velocity and the no-slip condition:

$$\mathbf{v}_M^e = \mathbf{v}_P = 0.$$

Formula (11) gives:

$$0 = \frac{d}{dt}\mathbf{v}_M^e = \mathbf{a}_M^e + \boldsymbol{\omega} \times \mathbf{v}_M^r, \quad \mathbf{a}_M^e = \mathbf{a}_P.$$

The theorem is proved.

7.4. Instantaneous Screw Motion of a Rigid Body. In conclusion, we note in connection with Corollary 1 that, in the general case, at each moment of time in a rigid body there exists a point B such that $\mathbf{v}_B \parallel \boldsymbol{\omega}$. If $\boldsymbol{\omega} \neq 0$, these points form a straight line parallel to $\boldsymbol{\omega}$. Point B can be found from the formula

$$\mathbf{AB} = \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{|\boldsymbol{\omega}|^2}.$$

If the body undergoes planar motion (i.e., the velocities of all its points are parallel to a fixed plane), then its angular velocity is directed perpendicular to this plane, and point B is the instantaneous center of zero velocity ($\mathbf{v}_B = 0$).

7.5. Rotation Matrix of a Rigid Body. Let $\mathbf{u}$ be an arbitrary vector attached to a rigid body moving with angular velocity $\boldsymbol{\omega}(t)$. Let $u(t)$ denote the coordinate column vector of $\mathbf{u}$ in the fixed system $Oxyz$. We have

$$u(t) = X(t)u_0, \quad u(0) = u_0,$$

where $X(t)$ is a certain 3×3 orthogonal matrix, $X(0) = E$.

Suppose that the system $Q\xi\eta\zeta$ attached to the rigid body coincides with the system $Oxyz$ at the initial moment of time.

In particular, this means that the coordinate column vector of $\mathbf{u}$ in the system $Q\xi\eta\zeta$ is equal to u_0 at all times. Moreover, if v is the coordinate column vector of some vector $\mathbf{v}$ in the system $Oxyz$, then $X^{-1}(t)v$ is the coordinate column vector of $\mathbf{v}$ in the system $Q\xi\eta\zeta$.

Formula

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \times \mathbf{u} \tag{12}$$

is invariant; it is a consequence of formula (3).

Suppose that the angular velocity vector $\boldsymbol{\omega}$ is given by its coordinate column vector ω in the system $Q\xi\eta\zeta$.

Then we can write formula (12) in coordinates of $Q\xi\eta\zeta$:

$$X^{-1}\dot{u}(t) = \omega \times u_0 = \Omega u_0,$$

where

$$\Omega = \begin{pmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{pmatrix}.$$

Finally, we obtain

$$X^{-1}\dot{X}u_0 = \Omega u_0 \implies \dot{X} = X\Omega.$$

References

- [1] N.V. Efimov, E.R. Rozendorn: Linear Algebra and Multidimensional Geometry. Nauka, 1970. (in Russian)