Lecture Notes on Kinematics

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For details see:

1) Ya.V. Tatarinov: Lectures on Classical Dynamics. MSU Press, 1984(in Russian);

2) S.V. Bolotin, A.V. Karapetyan, E.I. Kugushev, D.V. Treschev: Theoretical Mechanics. Moscow, "Academia", 2010(in Russian).

1. Velocity and Acceleration of a Point

Let us fix a positively oriented Cartesian coordinate system $Oxyz$ in the physical $\mathbb{R}^3$.

The law of motion of a point M with radius vector $\boldsymbol{r}_M(t) = \boldsymbol{OM}(t)$ is defined as the mapping

$$t \mapsto \boldsymbol{r}_M(t) = x(t)\boldsymbol{e}_x + y(t)\boldsymbol{e}_y + z(t)\boldsymbol{e}_z, \quad x(\cdot), y(\cdot), z(\cdot) \in C^2[0,T].$$

Here $\boldsymbol{e}_x$ and so on denote the unit basis vectors of the corresponding axes.

The trajectory of a point M is the set of points

$$\{\boldsymbol{r}_M(t) \mid t \in [0,T]\} \subset \mathbb{R}^3,$$

which is typically a piecewise smooth curve.

The velocity and acceleration of the point M relative to $Oxyz$ are defined respectively as the vectors

$$\boldsymbol{v}_M = \frac{d\boldsymbol{r}_M}{dt} = \dot{\boldsymbol{r}}_M = \dot{x}(t)\boldsymbol{e}_x + \dot{y}(t)\boldsymbol{e}_y + \dot{z}(t)\boldsymbol{e}_z, \quad \boldsymbol{a}_M = \dot{\boldsymbol{v}}_M.$$

In the following, unless specified otherwise, velocities and accelerations are considered relative to the system $Oxyz$.

2. Kinematics of a Rigid Body

Definition 1. *Let $D \subset \mathbb{R}^3$ be a non-empty open set and consider a pair of functions*

$$\boldsymbol{r}_A \in C^2\big([0,T], \mathbb{R}^3\big), \quad U \in C^2\big([0,T], \mathrm{SO}(3)\big), \quad U(0) = E, \quad \boldsymbol{r}_A(0) \in \overline{D}.$$

By definition, a rigid body consists of points B moving according to the law

$$\boldsymbol{r}_B(t) = \boldsymbol{r}_A(t) + U(t)\big(\boldsymbol{r}_B(0) - \boldsymbol{r}_A(0)\big), \quad t \in [0,T], \tag{1}$$

where the initial condition $\boldsymbol{r}_B(0) \in \overline{D}$.

The law of motion of point B will not change if, instead of the radius vector $\boldsymbol{r}_A$ in formula (1), we take the radius vector of any other point belonging to the rigid body.

Definition 2. *A vector $\boldsymbol{u}(t)$ is said to be attached to (or embedded in) the rigid body if there exist two points of the rigid body P, N and a number λ such that $\boldsymbol{u}(t) = \lambda(\boldsymbol{r}_P(t) - \boldsymbol{r}_N(t)), \quad t \in [0,T]$.*

Theorem 1 (Euler's Formula). *There exists a unique (axial) vector* $\boldsymbol{\omega}$ *such that for any two points* A, B *of the rigid body, the following equality holds*

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{AB}, \quad \boldsymbol{AB} = \boldsymbol{r}_B - \boldsymbol{r}_A.$$

Definition 3. *The vector* $\boldsymbol{\omega}$ *is called the angular velocity of the rigid body.*
The vector $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ *is called the angular acceleration of the rigid body.*

Proof of Theorem 1. Differentiating equality (1), we find

$$\boldsymbol{v}_B(t) = \boldsymbol{v}_A(t) + \dot{U}(t)\big(\boldsymbol{r}_B(0) - \boldsymbol{r}_A(0)\big) = \boldsymbol{v}_A(t) + \dot{U}(t)U^{-1}(t)\big(\boldsymbol{r}_B(t) - \boldsymbol{r}_A(t)\big).$$

The operator $\Omega = \dot{U}U^{-1} = \dot{U}U^*$ is skew-symmetric:

$$UU^* = E, \quad \frac{d}{dt}(UU^*) = \dot{U}U^* + U\dot{U}^* = 0 \Longrightarrow \Omega^* = -\Omega.$$

From linear algebra [**1**], it is known that a skew-symmetric operator in $\mathbb{R}^3$ corresponds to a unique axial vector $\boldsymbol{\omega}$ such that

$$\Omega \boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}, \quad \forall \boldsymbol{x} \in \mathbb{R}^3.$$

Let us show that the angular velocity vector is uniquely defined. Assume the contrary: there exists another vector $\boldsymbol{\omega}'$ satisfying Euler's formula

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{\omega}' \times \boldsymbol{AB}$$

for any two points A, B of the rigid body. Subtracting one formula from the other gives:

$$(\boldsymbol{\omega}' - \boldsymbol{\omega}) \times \boldsymbol{AB} = 0.$$

It remains to choose a non-zero vector $\boldsymbol{AB}$ perpendicular to $\boldsymbol{\omega}' - \boldsymbol{\omega}$. Contradiction.
The theorem is proved.

Corollary 1. *Suppose that at some moment in time* $\boldsymbol{v}_A = 0$ *with* $\boldsymbol{\omega} \neq 0$*. Then the set of points of the rigid body whose instantaneous velocities are equal to zero forms a straight line passing through point* A *parallel to the vector* $\boldsymbol{\omega}$*. This line is called the instantaneous axis of rotation. The velocities of all other points of the body at this moment are perpendicular to this axis.*

The motion of a rigid body is called translational if the velocities of all its points are equal. The motion of a rigid body is translational if and only if $\boldsymbol{\omega} = 0$.
Let us attach a positively oriented orthonormal frame $\boldsymbol{e}_1(t)\boldsymbol{e}_2(t)\boldsymbol{e}_3(t)$ to the rigid body.

Lemma 1. *The following formula holds*[1]

$$\boldsymbol{\omega} = (\dot{\boldsymbol{e}}_2, \boldsymbol{e}_3)\boldsymbol{e}_1 + (\dot{\boldsymbol{e}}_3, \boldsymbol{e}_1)\boldsymbol{e}_2 + (\dot{\boldsymbol{e}}_1, \boldsymbol{e}_2)\boldsymbol{e}_3. \tag{2}$$

Indeed, let us rewrite Euler's formula as follows

$$\frac{d}{dt}\boldsymbol{AB} = \boldsymbol{\omega} \times \boldsymbol{AB}. \tag{3}$$

In particular, *Poisson's formulas* hold:

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i, \quad i = 1, 2, 3. \tag{4}$$

Now, to verify the statement of the lemma, it is sufficient to substitute (4) into the right-hand side of (2) and use the properties of the triple product.
The lemma is proved.

[1] Note that formula (2) as such is not suitable as a definition of angular velocity, since its right-hand side a priori depends on the choice of the frame attached to the rigid body.

THEOREM 2 (Rivals' Formula). *The following formula holds*

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{\varepsilon} \times \boldsymbol{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{AB}).$$

Indeed, this formula is obtained by direct differentiation of Euler's formula using (3).

3. Rotation of a Rigid Body Around a Fixed Axis

Consider an important special case. Suppose that a rigid body rotates around a fixed axis Oz by an angle $\varphi = \varphi(t)$ such that for some positively oriented frame $\{\boldsymbol{e}_i\}$ attached to the rigid body, the following formulas hold: $\boldsymbol{e}_3 \equiv \boldsymbol{e}_z$,

$$\boldsymbol{e}_1 = \cos\varphi \boldsymbol{e}_x + \sin\varphi \boldsymbol{e}_y, \quad \boldsymbol{e}_2 = \cos\varphi \boldsymbol{e}_y - \sin\varphi \boldsymbol{e}_x.$$

Differentiating the first formula:

$$\dot{\boldsymbol{e}}_1 = \dot{\varphi}(-\sin\varphi \boldsymbol{e}_x + \cos\varphi \boldsymbol{e}_y) = \dot{\varphi}\boldsymbol{e}_2.$$

Similarly: $\dot{\boldsymbol{e}}_2 = -\dot{\varphi}\boldsymbol{e}_1, \quad \dot{\boldsymbol{e}}_3 = 0.$

Substituting these formulas into (2), we find

$$\boldsymbol{\omega} = \dot{\varphi}\boldsymbol{e}_3 = \dot{\varphi}\boldsymbol{e}_z. \tag{5}$$

4. Kinematics of Relative Motion

Suppose that a positively oriented Cartesian coordinate system $Q\xi\eta\zeta$ moves relative to $Oxyz$ with angular velocity $\boldsymbol{\omega}^e(t)$ and such that $\boldsymbol{OQ} = \boldsymbol{OQ}(t)$. Note that a Cartesian coordinate system can be considered as attached to some rigid body.

Let us introduce the notation

$$\boldsymbol{e}_1 = \boldsymbol{e}_\xi, \quad \boldsymbol{e}_2 = \boldsymbol{e}_\eta, \quad \boldsymbol{e}_3 = \boldsymbol{e}_\zeta.$$

Let $\boldsymbol{f}(t)$ be some time-dependent vector. We can write

$$\boldsymbol{f} = f_x(t)\boldsymbol{e}_x + f_y(t)\boldsymbol{e}_y + f_z(t)\boldsymbol{e}_z = f^i(t)\boldsymbol{e}_i. \tag{6}$$

The derivative of this vector with respect to the system $Oxyz$ will be considered, as above, to be the vector

$$\dot{\boldsymbol{f}} = \dot{f}_x\boldsymbol{e}_x + \dot{f}_y\boldsymbol{e}_y + \dot{f}_z\boldsymbol{e}_z;$$

and the derivative with respect to the system $Q\xi\eta\zeta$ to be the vector

$$\frac{\delta \boldsymbol{f}}{\delta t} := \dot{f}^i\boldsymbol{e}_i.$$

LEMMA 2. *The following formula holds*

$$\dot{\boldsymbol{f}} = \frac{\delta \boldsymbol{f}}{\delta t} + \boldsymbol{\omega}^e \times \boldsymbol{f}. \tag{7}$$

Indeed, this formula is obtained by direct differentiation of equality (6) using Poisson's formulas.

Let us investigate the relative motion of a point M.

DEFINITION 4. *The relative velocity of point M is defined as its velocity relative to the system $Q\xi\eta\zeta$:*

$$\boldsymbol{v}^r = \frac{\delta \boldsymbol{QM}}{\delta t}.$$

The transport (or frame) velocity of point M is defined as the velocity of the point of the system $Q\xi\eta\zeta$ that coincides with M at the given moment of time. It is denoted by $\boldsymbol{v}^e$.

The relative acceleration of point M is defined as its acceleration relative to the system $Q\xi\eta\zeta$:

$$\boldsymbol{a}^r = \frac{\delta \boldsymbol{v}^r}{\delta t}.$$

The transport (or frame) acceleration of point M is defined as the acceleration of the point of the system $Q\xi\eta\zeta$ that coincides with M at the given moment of time. It is denoted by $\boldsymbol{a}^e$.

The Coriolis acceleration of point M is defined as the vector $\boldsymbol{a}^c = 2(\boldsymbol{\omega}^e \times \boldsymbol{v}^r)$.

Note that in general $\boldsymbol{a}^e \neq \dot{\boldsymbol{v}}^e$.

THEOREM 3. *The following formulas hold*

$$\boldsymbol{v}_M = \boldsymbol{v}^r + \boldsymbol{v}^e, \tag{8}$$

$$\boldsymbol{a}_M = \boldsymbol{a}^r + \boldsymbol{a}^e + \boldsymbol{a}^c. \tag{9}$$

Proof. Let us differentiate the identity

$$\boldsymbol{OM} = \boldsymbol{OQ} + \boldsymbol{QM},$$

using (7):

$$\boldsymbol{v}_M = \boldsymbol{v}_Q + \frac{\delta \boldsymbol{QM}}{\delta t} + \boldsymbol{\omega}^e \times \boldsymbol{QM} = \boldsymbol{v}_Q + \boldsymbol{v}^r + \boldsymbol{\omega}^e \times \boldsymbol{QM}. \tag{10}$$

By Euler's formula

$$\boldsymbol{v}_Q + \boldsymbol{\omega}^e \times \boldsymbol{QM} = \boldsymbol{v}^e.$$

Formula (8) is proved.

Let us verify formula (9). Differentiating (10):

$$\boldsymbol{a}_M = \boldsymbol{a}_Q + \frac{\delta \boldsymbol{v}^r}{\delta t} + \boldsymbol{\omega}^e \times \boldsymbol{v}^r + \boldsymbol{\varepsilon}^e \times \boldsymbol{QM} + \boldsymbol{\omega}^e \times \Big(\frac{\delta \boldsymbol{QM}}{\delta t} + \boldsymbol{\omega}^e \times \boldsymbol{QM}\Big), \quad \boldsymbol{\varepsilon}^e = \dot{\boldsymbol{\omega}}^e.$$

By Rivals' formula

$$\boldsymbol{a}_Q + \boldsymbol{\varepsilon}^e \times \boldsymbol{QM} + \boldsymbol{\omega}^e \times (\boldsymbol{\omega}^e \times \boldsymbol{QM}) = \boldsymbol{a}^e.$$

The theorem is proved.

Suppose that a rigid body moves relative to the system $Q\xi\eta\zeta$ with angular velocity $\boldsymbol{\omega}^r$, and relative to the system $Oxyz$ with angular velocity $\boldsymbol{\omega}$.

THEOREM 4. *The following formula holds $\boldsymbol{\omega} = \boldsymbol{\omega}^e + \boldsymbol{\omega}^r$.*

Proof. Take arbitrary points A, B of the rigid body. Using the velocity addition formula (8):

$$\boldsymbol{v}_B = \boldsymbol{v}^r_B + \boldsymbol{v}^e_B.$$

Applying Euler's formula to the rigid body, we find:

$$\boldsymbol{v}^r_B = \boldsymbol{v}^r_A + \boldsymbol{\omega}^r \times \boldsymbol{AB}.$$

Applying Euler's formula to the system $Q\xi\eta\zeta$, we find:

$$\boldsymbol{v}^e_B = \boldsymbol{v}^e_A + \boldsymbol{\omega}^e \times \boldsymbol{AB}.$$

Putting it all together, we have:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + (\boldsymbol{\omega}^r + \boldsymbol{\omega}^e) \times \boldsymbol{AB}.$$

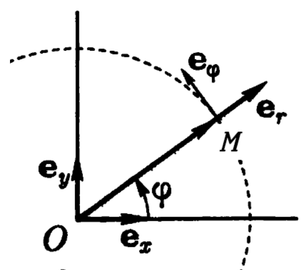
By the uniqueness of the angular velocity, we obtain the required result.

The theorem is proved.

5. Velocity and Acceleration of a Point in Polar Coordinates

Let a point M move in the Oxy plane, where polar coordinates are introduced:

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$



By formula (5), the angular velocity of the frame $\boldsymbol{e}_r\boldsymbol{e}_\varphi\boldsymbol{e}_z$ is equal to $\boldsymbol{\omega} = \dot\varphi\boldsymbol{e}_z$. From which, by Poisson's formulas:

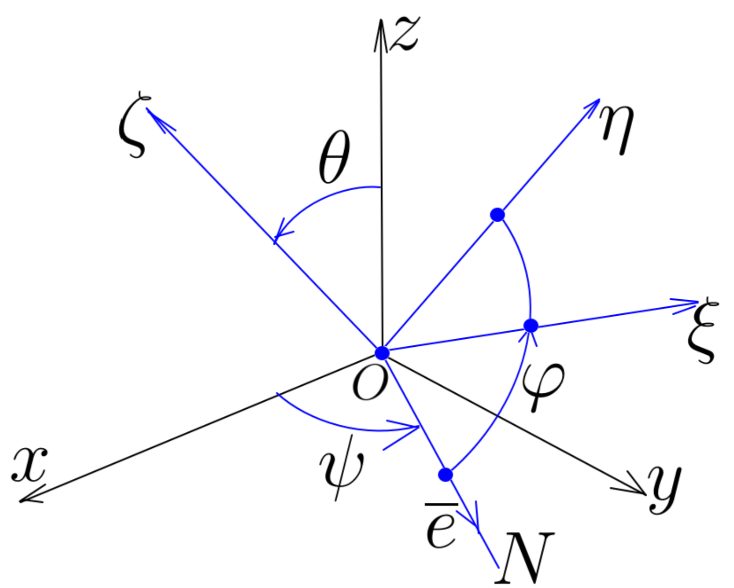
$$\dot{\boldsymbol{e}}_r = \boldsymbol{\omega}\times\boldsymbol{e}_r = \dot\varphi\boldsymbol{e}_\varphi, \quad \dot{\boldsymbol{e}}_\varphi = \boldsymbol{\omega}\times\boldsymbol{e}_\varphi = -\dot\varphi\boldsymbol{e}_r.$$

Now the velocity and acceleration of point M in polar coordinates are calculated by differentiating the radius vector $\boldsymbol{OM} = r\boldsymbol{e}_r$:

$$\boldsymbol{v}_M = \frac{d}{dt}\boldsymbol{OM} = \dot r\boldsymbol{e}_r + r\dot\varphi\boldsymbol{e}_\varphi, \quad \boldsymbol{a}_M = \frac{d}{dt}\boldsymbol{v}_M = (\ddot r - r\dot\varphi^2)\boldsymbol{e}_r + (r\ddot\varphi + 2\dot r\dot\varphi)\boldsymbol{e}_\varphi.$$

6. Euler Angles and Euler's Kinematic Formulas

The Euler angles determine the position of a rigid body with a fixed point O relative to the fixed frame $Oxyz$. Let us attach a positively oriented orthonormal frame $O\xi\eta\zeta$ to the rigid body.



The line ON (the line of nodes) lies in the intersection of the planes $O\xi\eta$ and Oxy. It is assumed that these planes do not coincide. The Euler angles

$$\theta\in(0,\pi), \quad \varphi \pmod{2\pi}, \quad \psi \pmod{2\pi}$$

are shown in the figure.

Let I denote the coordinate system associated with the axes ON and Oz;

Let II denote the coordinate system associated with the axes ON and $O\zeta$;

By formula (5), the angular velocity of I relative to $Oxyz$ is

$$\boldsymbol{\omega}_1 = \dot\psi \boldsymbol{e}_z;$$

the angular velocity of II relative to I is

$$\boldsymbol{\omega}_2 = \dot\theta \boldsymbol{e}, \quad \boldsymbol{e} = \cos\varphi \boldsymbol{e}_\xi - \sin\varphi \boldsymbol{e}_\eta;$$

the angular velocity of $O\xi\eta\zeta$ relative to II is

$$\boldsymbol{\omega}_3 = \dot\varphi \boldsymbol{e}_\zeta.$$

By Theorem 4, the angular velocity of the frame $O\xi\eta\zeta$ relative to $Oxyz$ is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3.$$

Note that the plane $O\zeta z$ is perpendicular to the line ON, therefore

$$\boldsymbol{e}_z = \cos\theta \boldsymbol{e}_\zeta + \sin\theta(\boldsymbol{e}_\zeta \times \boldsymbol{e}).$$

Thus,

$$\begin{aligned}\boldsymbol{\omega} &= (\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi)\boldsymbol{e}_\xi \\ &\quad + (\dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi)\boldsymbol{e}_\eta \\ &\quad + (\dot\psi \cos\theta + \dot\varphi)\boldsymbol{e}_\zeta.\end{aligned}$$

7. Appendix

7.1. A Simply Useful Fact. The projections of the velocities of two points A and B of a rigid body onto the straight line connecting them are equal:

$$(\boldsymbol{v}_A, \boldsymbol{e}) = (\boldsymbol{v}_B, \boldsymbol{e}), \quad \boldsymbol{e} = \frac{\boldsymbol{AB}}{|\boldsymbol{AB}|}.$$

7.2. Kinematics of the Frenet Frame. Let a curve γ be given parametrically by the radius vector $\boldsymbol{r} = \boldsymbol{r}(s)$, where s is the arc length parameter:

$$\frac{d}{ds}\boldsymbol{r} = \boldsymbol{r}', \quad |\boldsymbol{r}'| \equiv 1.$$

Suppose that the vectors $\boldsymbol{\tau} = \boldsymbol{r}'$ and $\boldsymbol{r}''$ are linearly independent. Note that $\boldsymbol{r}' \perp \boldsymbol{r}''$.

Define the curvature of the curve by the formula $k(s) = |\boldsymbol{r}''|$.

The Frenet formulas:

$$\boldsymbol{\tau}'(s) = k(s)\boldsymbol{n}(s), \quad \boldsymbol{n}'(s) = -k(s)\boldsymbol{\tau}(s) - \varkappa(s)\boldsymbol{b}(s), \quad \boldsymbol{b}'(s) = \varkappa(s)\boldsymbol{n}(s).$$

Here $\boldsymbol{b} = \boldsymbol{\tau} \times \boldsymbol{n}$.

Let us decompose the radius vector with respect to the Frenet frame: $\boldsymbol{r} = x\boldsymbol{\tau} + y\boldsymbol{n} + z\boldsymbol{b}$.

Differentiating this formula and substituting the result into the equality $\boldsymbol{r}' = \boldsymbol{\tau}$, we obtain useful relations:

$$x' = ky + 1, \quad y' = -kx - \varkappa z, \quad z' = \varkappa y.$$

Suppose that the Frenet frame $M\boldsymbol{\tau}\boldsymbol{n}\boldsymbol{b}$ is constructed at point M, which moves along the curve according to the law $\boldsymbol{r}_M = \boldsymbol{r}(s(t))$. Then the angular velocity of the Frenet frame is calculated by formula (2):

$$\boldsymbol{\omega} = \dot{s}(-\varkappa\boldsymbol{\tau} + k\boldsymbol{b}), \quad \Big(\frac{d}{dt} = \dot{s}\frac{d}{ds}\Big).$$

In particular, if the curve is planar ($\varkappa = 0$), then from formula (5) we find: $k(s) = \varphi'$.

The velocity and acceleration of point M in such motion are given by the formulas

$$\boldsymbol{v}_M = \dot{\boldsymbol{r}}_M = \boldsymbol{r}'\dot{s} = \dot{s}\boldsymbol{\tau}, \quad \boldsymbol{a}_M = \ddot{s}\boldsymbol{\tau} + \dot{s}^2 k\boldsymbol{n}.$$

7.3. Rolling Without Slipping. Let a convex rigid body roll without slipping with angular velocity $\boldsymbol{\omega}$ on a fixed plane.

Let P denote the point of the body that touches the plane at the given moment of time. Rolling without slipping means by definition that $\boldsymbol{v}_P = 0$.

Let M denote the geometric point of contact between the body and the plane. The point of contact traces a curve on the plane during the motion.

THEOREM 5. *The following formula holds* $\boldsymbol{a}_P = -\boldsymbol{\omega} \times \boldsymbol{v}_M$.

Proof. Let us note the following formula

$$\frac{d\boldsymbol{v}^e}{dt} = \boldsymbol{a}^e + \boldsymbol{\omega}^e \times \boldsymbol{v}^r. \tag{11}$$

It is obtained by differentiating formula (8):

$$\boldsymbol{a}_M = \frac{\delta\boldsymbol{v}^r}{\delta t} + \boldsymbol{\omega}^e \times \boldsymbol{v}^r + \frac{d\boldsymbol{v}^e}{dt} = \boldsymbol{a}^r + \boldsymbol{\omega}^e \times \boldsymbol{v}^r + \frac{d\boldsymbol{v}^e}{dt}$$

and comparing this result with (9).

Let us attach the system $Q\xi\eta\zeta$ to the body. Then

$$\boldsymbol{v}_M = \boldsymbol{v}^r_M + \boldsymbol{v}^e_M = \boldsymbol{v}^r_M.$$

Here we used the definition of the transport velocity and the no-slip condition:

$$\boldsymbol{v}^e_M = \boldsymbol{v}_P = 0.$$

Formula (11) gives:

$$0 = \frac{d}{dt}\boldsymbol{v}^e_M = \boldsymbol{a}^e_M + \boldsymbol{\omega} \times \boldsymbol{v}^r_M, \quad \boldsymbol{a}^e_M = \boldsymbol{a}_P.$$

The theorem is proved.

7.4. Instantaneous Screw Motion of a Rigid Body. In conclusion, we note in connection with Corollary 1 that, in the general case, at each moment of time in a rigid body there exists a point B such that $\boldsymbol{v}_B \parallel \boldsymbol{\omega}$. If $\boldsymbol{\omega} \neq 0$, these points form a straight line parallel to $\boldsymbol{\omega}$. Point B can be found from the formula

$$\boldsymbol{AB} = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_A}{|\boldsymbol{\omega}|^2}.$$

If the body undergoes planar motion (i.e., the velocities of all its points are parallel to a fixed plane), then its angular velocity is directed perpendicular to this plane, and point B is the instantaneous center of zero velocity ($\boldsymbol{v}_B = 0$).

7.5. Rotation Matrix of a Rigid Body. Let $\boldsymbol{u}$ be an arbitrary vector attached to a rigid body moving with angular velocity $\boldsymbol{\omega}(t)$. Let $u(t)$ denote the coordinate column vector of $\boldsymbol{u}$ in the fixed system $Oxyz$. We have

$$u(t) = X(t)u_0, \quad u(0) = u_0,$$

where $X(t)$ is a certain 3×3 orthogonal matrix, $X(0) = E$.

Suppose that the system $Q\xi\eta\zeta$ attached to the rigid body coincides with the system $Oxyz$ at the initial moment of time.

In particular, this means that the coordinate column vector of $\boldsymbol{u}$ in the system $Q\xi\eta\zeta$ is equal to u_0 at all times. Moreover, if v is the coordinate column vector of some vector $\boldsymbol{v}$ in the system $Oxyz$, then $X^{-1}(t)v$ is the coordinate column vector of $\boldsymbol{v}$ in the system $Q\xi\eta\zeta$.

Formula

$$\dot{\boldsymbol{u}} = \boldsymbol{\omega} \times \boldsymbol{u} \tag{12}$$

is invariant; it is a consequence of formula (3).

Suppose that the angular velocity vector $\boldsymbol{\omega}$ is given by its coordinate column vector ω in the system $Q\xi\eta\zeta$.

Then we can write formula (12) in coordinates of $Q\xi\eta\zeta$:

$$X^{-1}\dot{u}(t) = \omega \times u_0 = \Omega u_0,$$

where

$$\Omega = \begin{pmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{pmatrix}.$$

Finally, we obtain

$$X^{-1}\dot{X}u_0 = \Omega u_0 \Longrightarrow \dot{X} = X\Omega.$$

References

[1] N.V. Efimov, E.R. Rozendorn: Linear Algebra and Multidimensional Geometry. Nauka, 1970. (in Russian)