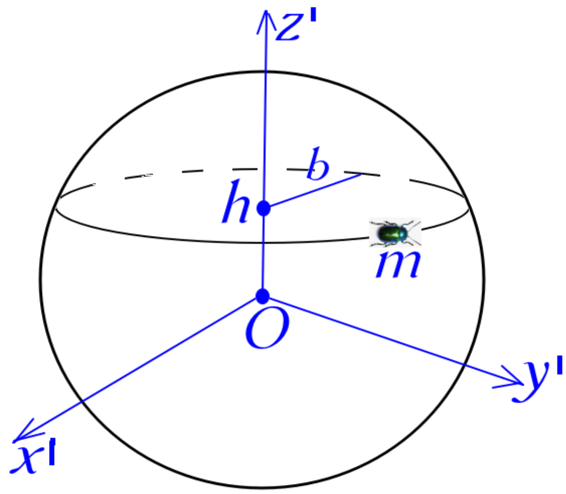

Let us denote by T the time from the start of the beetle's motion until it stops.

Let $Ox'y'z'$ be a ball-fixed coordinate frame such that the beetle moves in the plane $z' = h, \quad (h^2 + b^2 = R^2)$. Then the position vector of the beetle is given by the formula

$$\boldsymbol{r}(t) = h\boldsymbol{e}'_z + b(\cos\alpha\boldsymbol{e}'_x + \sin\alpha\boldsymbol{e}'_y).$$

According to the condition, the function $\alpha = \alpha(t)$ strictly increases on the interval $[0, T]$ and is such that

$$\alpha(0) = 0, \quad \dot\alpha(0) = \dot\alpha(T) = 0, \quad \alpha(T) = 2\pi.$$

The angular momentum of the system about the point O is obviously conserved:

$$\boldsymbol{K}_O = J\boldsymbol{\omega} + m(\boldsymbol{r} \times \boldsymbol{v}) = 0. \qquad (1)$$

Here $\boldsymbol{\omega}$ is the angular velocity of the ball relative to the laboratory coordinate system, and $\boldsymbol{v}$ is the velocity of the beetle.

Since the position vector of the beetle is naturally defined in the system attached to the ball, it is convenient to expand the absolute velocity of the beetle using the velocity addition theorem:

$$\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{v}_e,$$

where

$$\boldsymbol{v}_e = \boldsymbol{\omega} \times \boldsymbol{r}$$

is the transport velocity of the beetle, and

$$\boldsymbol{v}_r = b\dot\alpha(-\sin\alpha\boldsymbol{e}'_x + \cos\alpha\boldsymbol{e}'_y)$$

is its relative velocity.

Substituting these formulas into equation (1) and solving the resulting linear algebraic equation for the vector $\boldsymbol{\omega}$, we find:

$$\boldsymbol{\omega} = -\frac{m}{J + mR^2}(\boldsymbol{r} \times \boldsymbol{v}_r), \qquad (2)$$

where

$$\boldsymbol{r} \times \boldsymbol{v}_r = \dot\alpha\big(b^2\boldsymbol{e}'_z - hb\cos\alpha\boldsymbol{e}'_x - hb\sin\alpha\boldsymbol{e}'_y\big). \qquad (3)$$

Formula (2) reflects an important fact: at the very moment the beetle stops on the ball, the ball also stops rotating.

Another important fact following from formulas (2) and (3) is as follows: the angular velocity vector of the ball, defined in the coordinate system attached to the ball, can be represented as

$$\boldsymbol{\omega} = \dot{\alpha}\boldsymbol{w}(\alpha),$$

where

$$\boldsymbol{w} = -\frac{m\big(b^2\boldsymbol{e}'_z - hb\cos\alpha\boldsymbol{e}'_x - hb\sin\alpha\boldsymbol{e}'_y\big)}{J + mR^2}. \qquad (4)$$

And this is where the most interesting part begins. Let us introduce a laboratory coordinate system $Oxyz$ such that at $t = 0$ it coincides with the system $Ox'y'z'$.

Let us denote by $X(t)$ a 3×3 matrix which is the solution to the following Cauchy problem for a matrix differential equation:

$$\dot{X} = \dot{\alpha}(t)XW(\alpha), \quad X(0) = I, \qquad (5)$$

where

$$W = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix},$$

and w_x, w_y, w_z are the components of the vector $\boldsymbol{w}$ (see formula (4)) in the $Ox'y'z'$ system.

The geometrical meaning of the operator $X(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is as follows. Suppose a vector $\boldsymbol{u}$ is rigidly attached to the ball. Its coordinates relative to the $Ox'y'z'$ system do not change, but relative to the $Oxyz$ system they do, since this vector rotates along with the ball. That is, an observer in the $Oxyz$ system sees the time dependence of the coordinates of the vector $\boldsymbol{u}$: $\boldsymbol{u} = \boldsymbol{u}(t)$.

In theoretical mechanics textbooks, it is proven that

$$\boldsymbol{u}(t) = X(t)\boldsymbol{u}(0).$$

In this formula, $\boldsymbol{u}(t)$ and $\boldsymbol{u}(0)$ are the coordinate columns of the vector $\boldsymbol{u}$ in the $Oxyz$ system at the corresponding moments in time.

Thus, $X(T)$ is the rotation matrix of the ball. This matrix contains the answer to the problem's question.

Let us change the variable $t \mapsto \alpha$ in (5):

$$\frac{d}{d\alpha}\tilde{X} = \tilde{X}W(\alpha), \quad \tilde{X}(0) = I. \qquad (6)$$

The solution to this system is the matrix $\tilde{X}(\alpha)$; accordingly, the required rotation matrix of the ball is $\tilde{X}(2\pi)$, $\quad \tilde{X}(2\pi) = X(T)$.

An important conclusion from this observation is that the rotation matrix of the ball is completely independent of the specific law of motion of the beetle $\alpha = \alpha(t)$.

It is highly unlikely that system (6) can be integrated in closed form. This question requires a separate study. However, the rotation matrix of the ball can always be found numerically by solving problem (6) on the interval $\alpha \in [0, 2\pi]$.