Faá di Bruno's formula for the kth derivative of the composition of two functions is this:

$$(f \circ g)^{(k)}(x) = \sum_{\pi \in \Pi} f^{(|\pi|)}(g(x)) \cdot \prod_{B \in \pi} g^{(|B|)}(x)$$

where

- π runs through the set Π of all partitions of the set $\{1, ..., k\}$,
- "$B \in \pi$" means the variable B runs through the list of all the blocks of the partition π, and
- $|A|$ denotes the cardinality of the set A (so that $|\pi|$ is the number of blocks in the partition π and $|B|$ is the size of the block B).

To find the kth derivative of the composition of n functions, we apply Faá di Bruno's formula repeatedly. Let Π_1 be the set of partitions of the set $\{1, ..., k\}$ and for $j > 1$ let Π_j be the set of partitions of $\{1, ..., |B_{j-1}|\}$ where B_j is a variable that runs through the list of all blocks of the partition π_j. π_j runs through the set Π_j. In the first step, we apply Faá di Bruno's formula with $f = f_1$ and $g = f_2 \circ \cdots \circ f_n$:

$$(f_1 \circ (f_2 \circ \cdots \circ f_n))^{(k)}(x) = \sum_{\pi_1 \in \Pi_1} f_1^{(|\pi_1|)}((f_2 \circ \cdots \circ f_n)(x)) \cdot \prod_{B_1 \in \pi_1} (f_2 \circ \cdots \circ f_n)^{(|B_1|)}(x).$$

Now to find $(f_2 \circ \cdots \circ f_n)^{(|B_1|)}(x)$, we apply Faá di Bruno's formula again:

$$(f_2 \circ \cdots \circ f_n)^{(|B_1|)}(x) = \sum_{\pi_2 \in \Pi_2} f_2^{(|\pi_2|)}((f_3 \circ \cdots \circ f_n)(x)) \cdot \prod_{B_2 \in \pi_2} (f_3 \circ \cdots \circ f_n)^{(|B_2|)}(x).$$

In general, for $j < n - 1$,

$$(f_j \circ \cdots \circ f_n)^{(|B_{j-1}|)}(x) = \sum_{\pi_j \in \Pi_j} f_j^{(|\pi_j|)}((f_{j+1} \circ \cdots \circ f_n)(x)) \cdot \prod_{B_j \in \pi_j} (f_{j+1} \circ \cdots \circ f_n)^{(|B_j|)}(x).$$

Finally, for $j = n - 1$ we have

$$(f_{n-1} \circ f_n)^{(|B_{n-2}|)}(x) = \sum_{\pi_{n-1} \in \Pi_{n-1}} f_{n-1}^{(|\pi_{n-1}|)}(f_n(x)) \cdot \prod_{B_{n-1} \in \pi_{n-1}} f_n^{(|B_{n-1}|)}(x).$$

Putting it all together, we get a formula for $(f_1 \circ \cdots \circ f_n)^{(k)}(x)$.

In the special case when $f_j = g$ for all j, we get a formula for the kth derivative of the nth iterate of g.

$$\left(g^{n}\right)^{(k)}(x) = \sum_{\pi_1 \in \Pi_1} g^{(|\pi_1|)}\left(g^{n-1}(x)\right) \cdot \prod_{B_1 \in \pi_1} \left(g^{n-1}\right)^{(|B_1|)}(x)$$

where

$$\left(g^{n-1}\right)^{(|B_1|)}(x) = \sum_{\pi_2 \in \Pi_2} g^{(|\pi_2|)}\left(g^{n-2}(x)\right) \cdot \prod_{B_2 \in \pi_2} \left(g^{n-2}\right)^{(|B_2|)}(x)$$

and for $j < n-2$,

$$\left(g^{n-j}\right)^{(|B_j|)}(x) = \sum_{\pi_{j+1} \in \Pi_{j+1}} g^{(|\pi_{j+1}|)}\left(g^{n-j-1}(x)\right) \cdot \prod_{B_{j+1} \in \pi_{j+1}} \left(g^{n-j-1}\right)^{(|B_{j+1}|)}(x).$$

Finally, for $j = n-2$, we have

$$\left(g^{2}\right)^{(|B_{n-2}|)}(x) = \sum_{\pi_{n-1} \in \Pi_{n-1}} g^{(|\pi_{n-1}|)}(g(x)) \cdot \prod_{B_{n-1} \in \pi_{n-1}} g^{(|B_{n-1}|)}(x).$$

Suppose further that p is a fixed point of g. Then

$$\left(g^{n}\right)^{(k)}(p) = \sum_{\pi_1 \in \Pi_1} g^{(|\pi_1|)}(p) \cdot \prod_{B_1 \in \pi_1} \left(g^{n-1}\right)^{(|B_1|)}(p)$$

where

$$\left(g^{n-1}\right)^{(|B_1|)}(p) = \sum_{\pi_2 \in \Pi_2} g^{(|\pi_2|)}(p) \cdot \prod_{B_2 \in \pi_2} \left(g^{n-2}\right)^{(|B_2|)}(p)$$

and for $j < n-2$,

$$\left(g^{n-j}\right)^{(|B_j|)}(p) = \sum_{\pi_{j+1} \in \Pi_{j+1}} g^{(|\pi_{j+1}|)}(p) \cdot \prod_{B_{j+1} \in \pi_{j+1}} \left(g^{n-j-1}\right)^{(|B_{j+1}|)}(p).$$

Finally, for $j = n-2$, we have

$$\left(g^{2}\right)^{(|B_{n-2}|)}(p) = \sum_{\pi_{n-1} \in \Pi_{n-1}} g^{(|\pi_{n-1}|)}(p) \cdot \prod_{B_{n-1} \in \pi_{n-1}} g^{(|B_{n-1}|)}(p).$$