$$τ\frac{d^{2}C\_{AM}\*}{dt^{2}}+ 2ζτ\frac{dC\_{AM}\*}{dt}+ C\_{AM}\*= K\_{P1}K\_{M}C\_{A01}\*+ K\_{P2}K\_{M}C\_{A02}\*$$

To solve this equation analytically a Laplace transform must be applied. It is impossible to transform the RHS until CA02\*and CA01\* are known so that is done later.

$$τ\frac{d^{2}C\_{AM}\*}{dt^{2}}= τ^{2}[s^{2}G\left(s\right)- sg\left(0\right)- g^{'}(0)]$$

$$2ζτ\frac{dC\_{AM}\*}{dt}= 2ζτ[sG\left(s\right)- g\left(0\right)]$$

$$C\_{AM}\* = G(s)$$

The next step is to work out what the value of g(0) and g’(0) is, i.e the value of CAM\* and its first derivative at t=0. The system is initially at steady state after running for an extended period of time. It is therefore reasonable to assume that the value of CA is equal to the set point CAs, if no disturbances have occurred, and therefore that CAM = CAMs. This makes CAM\* equal to zero.

Since the system is at steady state, it can be assumed that $\frac{dC\_{AM}\*}{dt} $is equal to zero at t=0 as well, because the values of the concentrations won’t be changing.

By this reasoning;

$$g\left(0\right)= 0$$

$$g^{'}\left(0\right)= 0$$

Adding each part together, the general transform is:

$$τ^{2}s^{2}G\left(s\right)+ 2ζτsG\left(s\right)+G\left(s\right)= L( K\_{P1}K\_{M}C\_{A01}\*+ K\_{P2}K\_{M}C\_{A02}\*)$$

1. At t=0, a step change of 3 kmolm-3 occurs in CA02\*. CA01\* is equal to zero. Other useful values are:

$τ= \sqrt{τ\_{m}τ\_{P}}= \sqrt{15×50}=5\sqrt{30} seconds$

$ζ=\frac{τ\_{P}+ τ\_{M}}{2√τ\_{P}τ\_{M}} = \frac{50 + 15}{2√50×15}= \frac{13√30}{30} $

$ K\_{P1}=0.4$

$K\_{P2}=0.1$

$K\_{M}=4 mA(kmolm^{-3})^{-1}$

$$τ^{2}s^{2}G\left(s\right)+ 2ζτsG\left(s\right)+G\left(s\right)=L\left(K\_{P1}K\_{M}C\_{A01}\*+K\_{P2}K\_{M}C\_{A02}\*\right)= \frac{K\_{P2}K\_{M}C\_{A02}\*}{s}$$

$$G\left(s\right)= \frac{K\_{P2}K\_{M}C\_{A02}\*}{s\left[τ^{2}s^{2}+ 2ζτs+1\right]}= \frac{As+B}{\left[τ^{2}s^{2}+ 2ζτs+1\right]}+\frac{C}{s}$$

$$\left(As+B\right)s+C\left(τ^{2}s^{2}+ 2ζτs+1\right)= K\_{P2}K\_{M}C\_{A02}\*$$

$$C= K\_{P2}K\_{M}C\_{A02}\*$$

$$B= -C2ζτ= -2K\_{P2}K\_{M}C\_{A02}\*ζτ$$

$$A= -Cτ^{2}= -K\_{P2}K\_{M}C\_{A02}\*τ^{2}$$

$$G\left(s\right)= K\_{P2}K\_{M}C\_{A02}\*[ \frac{1}{s} -\frac{τ^{2 }s+2ζτ}{\left[τ^{2}s^{2}+ 2ζτs+1\right]} ]$$

$$G\left(s\right)= K\_{P2}K\_{M}C\_{A02}\*[ \frac{1}{s}-\frac{s+2ζτ^{-1}}{(s+ ζτ^{-1})^{2}+ (τ^{-2}- (ζτ^{-1})^{2})} ] $$

The first half of the last equation is similar to the Laplace transform of a damped cos wave,

$$\frac{s+a}{(s+a)^{2}+ w^{2}}$$

 where:

$w^{2}= τ^{-2}- (ζτ^{-1})^{2}$ and

$a= ζτ^{-1}$, but it’s not quite right because what is actually happening is

$$\frac{s+2a}{(s+a)^{2}+ w^{2}}$$