Electronic Devices and Circuits

Engineering Sciences 154

Laplace Transforms and Circuit Analysis

$$F(s) = \int_{0_-}^{\infty} f(t)\exp(-st)\,dt$$

Laplace transform as a means of signal representation

Example 1 - unit step

$$F(s) = \int_{\tau_0}^{\infty} \{1\}\exp(-st)\,dt = \left\{-\frac{1}{s}\right\}\exp(-st)\Big|_{\tau_0}^{\infty} = \left\{-\frac{1}{s}\right\}\exp(-s\,\tau_0)$$

Example 2 - sinusoid

$$F(s) = \int_{0_-}^{\infty} \{\cos(\omega t - \varphi)\}\exp(-st)\,dt \qquad \Rightarrow \begin{cases} \cos(\omega t) & \text{when } \varphi = 0 \\ \sin(\omega t) & \text{when } \varphi = \dfrac{\pi}{2} \end{cases}$$

$$= \int_{0_-}^{\infty} \frac{1}{2}\left\{\exp[(j\,\omega - s)t - j\,\varphi] + \exp[-(j\,\omega + s)t + j\,\varphi]\right\} dt$$

$$= \frac{1}{2}\left\{\frac{1}{(j\,\omega - s)}\exp[(j\,\omega - s)t - j\,\varphi]\Big|_{0_-}^{\infty} + \frac{1}{-(j\,\omega + s)}\exp[-(j\,\omega + s)t + j\,\varphi]\Big|_{0_-}^{\infty}\right\}$$

$$= \frac{1}{2}\left\{\frac{-1}{(j\,\omega - s)}\exp(-j\,\varphi) + \frac{1}{(j\,\omega + s)}\exp(+j\,\varphi)\right\}$$

$$= \begin{cases} \dfrac{1}{2}\left[\dfrac{-1}{(j\,\omega - s)} + \dfrac{1}{(j\,\omega + s)}\right] = \dfrac{s}{s^2 + \omega^2} & \text{for } \varphi = 0 \\[2ex] \dfrac{j}{2}\left[\dfrac{1}{(j\,\omega - s)} + \dfrac{1}{(j\,\omega + s)}\right] = \dfrac{\omega}{s^2 + \omega^2} & \text{for } \varphi = \dfrac{\pi}{2} \end{cases}$$

Other Examples

Table of Laplace transforms of common wave forms
Table of Laplace transforms of basic functions
Table of Laplace transforms of trigonometric functions
Table of Laplace transform pairs

Laplace transform in circuit analysis

Property 1 - derivative

$$F^{deriv}(s) = \int_{0_-}^{\infty} \left\{ \frac{d}{dt} f(t) \right\} \exp(-st)\, dt$$

Integrate by parts

$$F^{deriv}(s) = \int_{0_-}^{\infty} \frac{d}{dt} \{ f(t) \exp(-st) \}\, dt - \int_{0_-}^{\infty} (-s) f(t) \exp(-st)\, dt$$

$$= s \int_{0_-}^{\infty} f(t) \exp(-st)\, dt - f(0_-) = s\, F(s) - f(0_-)$$

Property 2 - integral

$$F^{\text{integr}}(s) = \int_{0_-}^{\infty} \left\{ \left[\int_{0_-}^{t} \{ f(t')\, dt' \} \right] \right\} \exp(-st)\, dt$$

Integrate by parts

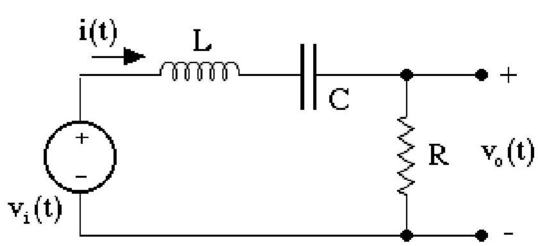
$$F^{\text{integr}}(s) = \int_{0_-}^{\infty} \frac{d}{dt} \left\{ \left[\int_{0_-}^{t} \{ f(t')\, dt' \} \right] \left[\frac{\exp(-st)}{-s} \right] \right\} dt - \int_{0_-}^{\infty} \left[\frac{\exp(-st)}{-s} \right] \frac{d}{dt} \left\{ \left[\int_{0_-}^{t} \{ f(t')\, dt' \} \right] \right\} dt$$

$$= \frac{1}{s} \int_{0_-}^{\infty} f(t) \exp(-st)\, dt = \frac{1}{s} F(s)$$

Other Properties

Table of Laplace transform properties

RLC Circuits
An Example of the Application of Laplace Transforms



Kirchoff's voltage law

$$v_i(t) = i(t)\,R + L\,\frac{d\,i(t)}{d\,t} + \frac{1}{C}\int_{0_-}^{t} i(t')\,d\,t'$$

Kirchoff's voltage law transformed

$$\begin{aligned} V_i(s) &= R\,I(s) + s\,L\,I(s) - L\,i(0_-) + \frac{1}{s\,C}\,I(s) \\ &= \left[R + s\,L + \frac{1}{s\,C}\right] I(s) - L\,i(0_-) \\ &= Z(s)\,I(s) - L\,i(0_-) \end{aligned}$$

where $Z(s) = R + s\,L + \frac{1}{s\,C}$

Example 1 - A series RLC circuit excited by a unit impulse function - that is to say

$$V_i(s) = 1$$

(The unit impulse always gives the "natural response" of any circuit. Natural as compared to "forced" response.)

$$I(s) = \frac{1}{Z(s)} = \frac{1}{R + s\,L + \frac{1}{s\,C}} = \frac{1}{L}\left\{\frac{s}{s^2 + s\,\frac{R}{L} + \frac{1}{LC}}\right\}$$

This is to be compared to the transform pair

$$f(t) = \sqrt{c^2 + d^2}\,\exp(-a\,t)\cos\left(\omega t - \tan^{-1}\frac{d}{c}\right)$$

$$F(s) = \frac{c(s+a) + d\,\omega}{(s+a)^2 + \omega^2}$$

See the table of [Laplace transform pairs]

Therefore, if

$$a = \frac{R}{2\,L} \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad d = -\frac{\omega}{a} = -\frac{2\,\omega\,L}{R}$$

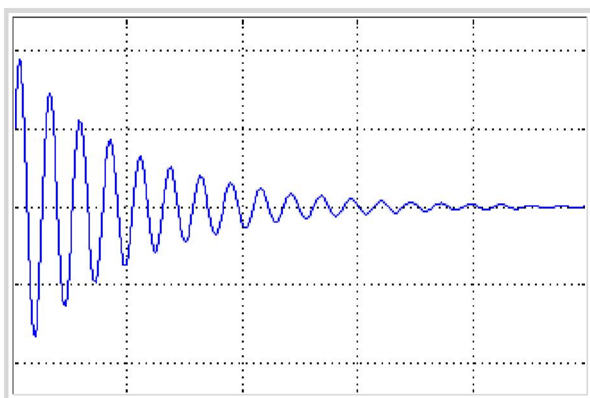
$$\omega^2 = \frac{1}{LC} - a^2 = \frac{1}{LC} - \left(\frac{R}{2\,L}\right)^2$$

$$\sqrt{c^2 + d^2} = \frac{\sqrt{a^2 + \omega^2}}{a} = \frac{2\,L}{R}\sqrt{\frac{1}{LC}}$$

$$\tan^{-1}\frac{d}{c} = -\tan^{-1}\left(\frac{2\,\omega\,L}{R}\right)$$

$$\boxed{\mathrm{v}_2(t) = \mathrm{i}(t)\,R = 2\sqrt{\frac{1}{LC}}\exp\left(-\frac{R}{2\,L}t\right)\cos\left\{\omega\,t + \tan^{-1}\left(\frac{2\,\omega\,L}{R}\right)\right\}}$$

where $\omega = \sqrt{\left[\frac{1}{LC} - \left(\frac{R}{2\,L}\right)^2\right]}$



Comparison - A series RC circuit excited by a unit impulse function - that is to say

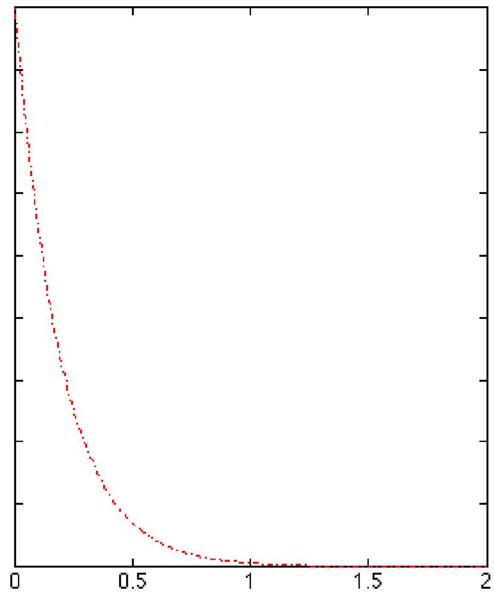
$$\mathrm{V}_{\mathrm{i}}(s) = 1$$

In general

$$\frac{V_{\mathrm{o}}(s)}{V_{\mathrm{i}}(s)} = \frac{R\,I(s)}{Z(s)\,I(s)} = \frac{s\,R\,C}{1 + s\,R\,C}$$

and in particular

$$V_{\mathrm{o}}(s) = \frac{s\,R\,C}{1 + s\,R\,C} = s\,\frac{1}{s + \frac{1}{R\,C}}$$



Example 2 - A series RLC circuit excited by a unit cosine function - that is to say

$$V_i(s) = \frac{s}{s^2 + \omega^2}$$

$$I(s) = \frac{1}{L}\left\{\frac{s}{s^2 + \omega^2}\right\}\left\{\frac{s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}\right\}$$

We need more methodology so go see Partial Fraction Expansions

See Symbolic Inverse Laplace Transform Applet

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