Line Integrals

Using an increment of length $dr = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$, we often encounter the line integral

$$\int_C \mathbf{V} \cdot d\mathbf{r}, \tag{1.101}$$

in which the integral is over some contour C that may be open (with starting point and ending point separated) or closed (forming a loop) instead of an interval of the x-axis. The Riemann integral is defined by subdividing the curve into ever smaller segments whose number grows indefinitely. The form [Eq. (1.101)] is exactly the same as that encountered when we calculate the work done by a force that varies along the path

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int F_x(x, y, z)\,dx + \int F_y(x, y, z)\,dy + \int F_z(x, y, z)\,dz, \tag{1.102}$$

that is, a sum of conventional integrals over intervals of one variable each. In this expression, $\mathbf{F}$ is the force exerted on a particle. In general, such integrals depend on the path except for conservative forces, whose treatment we postpone to Section 1.12.

EXAMPLE 1.9.1

Path-Dependent Work The force exerted on a body is $\mathbf{F} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$. The problem is to calculate the work done going from the origin to the point (1, 1),

$$W = \int_{0,0}^{1,1} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{1,1} (-y\,dx + x\,dy). \tag{1.103}$$

Separating the two integrals, we obtain

$$W = -\int_0^1 y\,dx + \int_0^1 x\,dy. \tag{1.104}$$

The first integral cannot be evaluated until we specify the values of y as x ranges from 0 to 1. Likewise, the second integral requires x as a function of y. Consider first the path shown in Fig. 1.27. Then

$$W = -\int_0^1 0\,dx + \int_0^1 1\,dy = 1 \tag{1.105}$$

because $y = 0$ along the first segment of the path and $x = 1$ along the second. If we select the path $[x = 0, 0 \le y \le 1]$ and $[0 \le x \le 1, y = 1]$, then Eq. (1.103) gives $W = -1$. For this force, the work done depends on the choice of path. ■

EXAMPLE 1.9.2

Line Integral for Work Find the work done going around a unit circle clockwise from 0 to $-\pi$ shown in Fig. 1.28 in the xy-plane doing work against a force field given by

$$\mathbf{F} = \frac{-\hat{\mathbf{x}}y}{x^2 + y^2} + \frac{\hat{\mathbf{y}}x}{x^2 + y^2}.$$

Figure 1.27

A Path of Integration

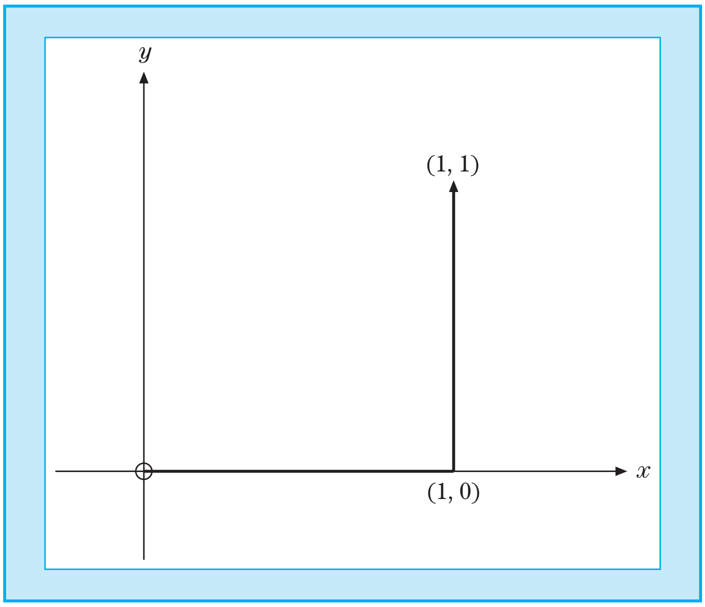
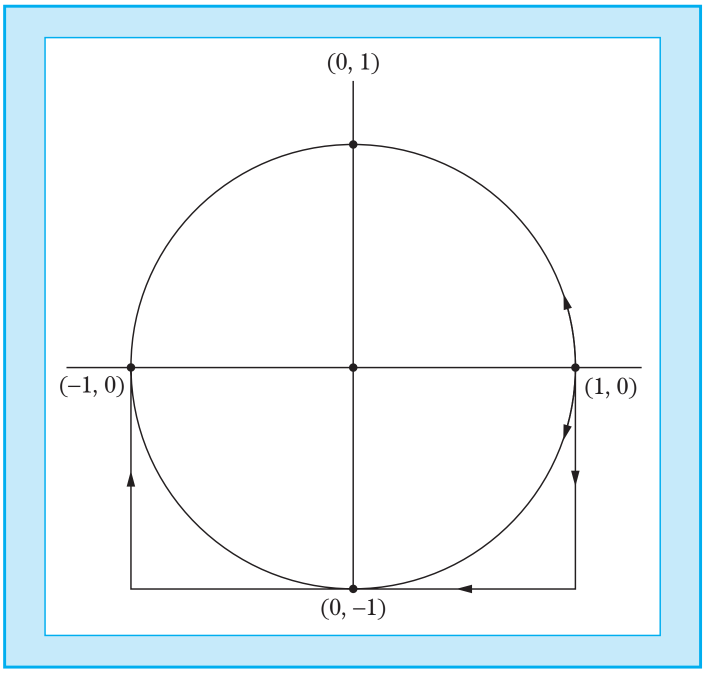


Figure 1.28

Circular and Square Integration Paths



Let us parameterize the circle C as $x = \cos\varphi$, $y = \sin\varphi$ with the polar angle φ so that $dx = -\sin\varphi d\varphi$, $dy = \cos\varphi d\varphi$. Then the force can be written as $\mathbf{F} = -\hat{\mathbf{x}}\sin\varphi + \hat{\mathbf{y}}\cos\varphi$. The work becomes

$$-\int_C \frac{xdy - ydx}{x^2 + y^2} = \int_0^{-\pi} (-\sin^2\varphi - \cos^2\varphi)\, d\varphi = \pi.$$

Here we spend energy. If we integrate anticlockwise from $\varphi = 0$ to π we find the value $-\pi$ because we are riding with the force. The work is path dependent, which is consistent with the physical interpretation that $\mathbf{F} \cdot d\mathbf{r} \sim xdy - ydx = L_z$ is proportional to the z-component of orbital angular momentum (involving circulation, as discussed in Section 1.7).

If we integrate along the square through the points $(\pm 1, 0)$, $(0, -1)$ surrounding the circle, we find for the clockwise lower half square path of Fig. 1.28

$$\begin{aligned}
-\int \mathbf{F} \cdot d\mathbf{r} &= -\int_0^{-1} F_y\, dy|_{x=1} - \int_1^{-1} F_x\, dx|_{y=-1} - \int_{-1}^{0} F_y\, dy|_{x=-1} \\
&= \int_0^1 \frac{dy}{1+y^2} + \int_{-1}^{1} \frac{dx}{x^2+(-1)^2} + \int_{-1}^{0} \frac{dy}{(-1)^2+y^2} \\
&= \arctan(1) + \arctan(1) - \arctan(-1) - \arctan(-1) \\
&= 4 \cdot \frac{\pi}{4} = \pi,
\end{aligned}$$

which is consistent with the circular path.

For the circular paths we used the $x = \cos\varphi$, $y = \sin\varphi$ parameterization, whereas for the square shape we used the standard definitions $y = f(x)$ or $x = g(y)$ of a curve, that is, $y = -1 = \text{const.}$ and $x = \pm 1 = \text{const.}$ We could have used the implicit definition $F(x, y) \equiv x^2 + y^2 - 1 = 0$ of the circle. Then the total variation

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 2x\, dx + 2y\, dy \equiv 0$$

so that

$$dy = -x\, dx/y \quad \text{with } y = -\sqrt{1-x^2}$$

on our half circle. The work becomes

$$\begin{aligned}
-\int_C \frac{x\, dy - y\, dx}{x^2+y^2} &= \int \left(\frac{x^2}{y} + y\right) dx = \int \frac{dx}{y} = \int_1^{-1} \frac{dx}{-\sqrt{1-x^2}} \\
&= \arcsin 1 - \arcsin(-1) = 2 \cdot \frac{\pi}{2} = \pi,
\end{aligned}$$

in agreement with our previous results. ■

EXAMPLE 1.9.3

Gravitational Potential If a force can be described by a scalar function V_G as $\mathbf{F} = -\nabla V_G(\mathbf{r})$ [Eq. (1.65)], everywhere we call V_G its potential in mechanics and engineering. Because the total variation $dV_G = \nabla V_G \cdot d\mathbf{r} = -\mathbf{F}_G \cdot d\mathbf{r}$ is the work done against the force along the path $d\mathbf{r}$, the integrated work along any path from the initial point $\mathbf{r}_0$ to the final point $\mathbf{r}$ is given by a line integral $\int_{\mathbf{r}_0}^{\mathbf{r}} dV_G = V_G(\mathbf{r}) - V_G(\mathbf{r}_0)$, the potential difference between the end points of