Theorem:

Let $\{\alpha_n\}_{n\in\omega}$ *be an infinite sequence of distinct subsets of* Ω. *Let* $\bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}\alpha_k) = \text{limit inf}\{\alpha_n\}$ & $\bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty}\alpha_k) = \text{lim sup}\{\alpha_n\}$.

Then $\text{limit inf}\{\alpha_n\} \subset \text{limit sup}\{\alpha_n\}$.

Proof:

$A_n = \bigcap_{k=n}^{\infty}\alpha_k$, $B_n = \bigcup_{k=n}^{\infty}\alpha_k$, $X = \text{limit inf}\{\alpha_n\}$, $Y = \text{lim sup}\{\alpha_n\}$

(1)we claim that $\bigcap_{k=1}^{n}\bigcup_{j=k}^{\infty}\alpha_j = \bigcup_{j=n}^{\infty}\alpha_j$

Proof: Let $C = B_1 \cap B_2$. $\forall x | x \in B_1 \wedge x \in \alpha_1,\ \overline{\alpha_1 \subset B_2} \rightarrow x \notin B_2 \Rightarrow x \notin C$

$(\forall n \geq 2)(\forall x) | x \in \alpha_n,\ \alpha_n \subset B_1 \wedge \alpha_n \subset B_2 \rightarrow x \in C$

Let $C_n = \bigcap_{k=1}^{n}\bigcup_{j=k}^{\infty}\alpha_j$. $\forall x | x \in B_n \wedge x \in \alpha_n,\ \overline{\alpha_n \subset B_{n+1}} \rightarrow x \notin B_{n+1} \Rightarrow x \notin C_n$

$(\forall n \geq 2)(\forall x) | x \in B_{n-1} \cap B_n \leftrightarrow x \in B_n \equiv \exists k \geq n | x \in \alpha_k$

Now $\text{lim sup}\{\alpha_n\} = \lim_{n\to\infty}\bigcap_{k=1}^{n}\bigcup_{j=k}^{\infty}\alpha_j = \lim_{n\to\infty}\bigcup_{k=n}^{\infty}\alpha_k = \lim_{n\to\infty}B_n$. So

$\forall\sigma | \sigma \in \lim_{n\to\infty}\bigcup_{k=n}^{\infty}\alpha_k \rightarrow (\forall n)\sigma \in B_n$ and since by the fact that $\forall n,\ \alpha_n$ are non-empty, distinct sets , $(\forall n)\, B_n \sim A_n \neq \emptyset \rightarrow A_n \subset B_n$. That is, $\{\forall B_n \exists x_n | x_n \in B_n \sim A_n\}$.

Choose $x_n | x_n \in \alpha_n \wedge \forall i > n,\ x_n \notin \alpha_i$

It follows from (1) that $(\forall n) B_n \supset B_{n+1}$ and so define the ordering $x_n < x_k \leftrightarrow B_n \subset B_k$ and $\phi | \phi(B_n) = x_n\ (\forall n)$. Then

$\forall n \exists \sigma | \sigma < x_n \Rightarrow \sigma \in \lim_{n\to\infty}\bigcup_{k=n}^{\infty}\alpha_k$ (*well ordering theorem* $\equiv$ *axiom of choice*).

Now since $\forall n,\ x_n \in (B_n \sim A_n,\ \lim_{n\to\infty}\{x_n\} \in \lim_{n\to\infty}(B_n \sim A_n) \neq \emptyset$ (*axiom of choice*).

So it follow that So it follows that $\exists x | x \in \text{lim sup} \wedge x \notin \text{lim inf}$.

From De Morgan's laws, $\text{lim inf}^{\,c} = [\bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty}\alpha_k)]^c = \bigcup_{n=1}^{\infty}(B_n)^c$ So

$x \in (B_n)^c | (\forall n)(1 \leq \exists r < n) \wedge x \in \alpha_r \wedge x \notin \alpha_k(\forall k \geq n) \rightarrow x \notin A_n \leftrightarrow x \in X^c$.

Hence, $x \in Y^c \rightarrow x \in X^c$. And by <u>modus tollens</u>, $(x \in Y^c \rightarrow x \in X^c) \Rightarrow$ $(\overline{x \in X^c} \equiv x \in X) \rightarrow (\overline{x \in Y^c} \equiv x \in Y)$. So $\forall x | x \in X \rightarrow x \in Y$.

Thus, $\forall x | x \in \text{lim inf} \rightarrow x \in \text{lim sup}$. $\therefore$ $\text{limit inf}\{\alpha_n\} \subset \text{limit sup}\{\alpha_n\}$.