The Lorentz Transformation

———deduced by logical arguments

[Equations regarding Time Dilation and Length Contraction in Special Relativity can be obtained by rigorous approaches through equations and the substitutions associated with it. But, they add very little visualization of the topic concerned. In that case, "Fundamentals of Physics (7E)" by Halliday/Resnick/Walker [Section 37.5 & 37.6] provided some logical arguments in establishing the above mentioned equations. There they used 'Mirror clocks', platforms, events to give an intriguing view about the concept of Relativity of time & length. But they did not do so in the case of 'The Lorentz Transformation'. Therefore, I intend to do that in the present article]

Using the assumptions of Special Relativity, the following two conclusions can be drawn:

1. Time Dilation:

If two events happen, one after another, at the same place with respect to a particular Inertial Reference frame, then the time interval between those two events, Δt_0 will be the lowest, compared to any other reference frames. In other words, for any reference frames with respect to which, those two events did not occur at the same place, we will find a larger time interval, Δt; where—

$$\Delta t = \gamma \Delta t_0 = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here,
v is the relative speed between the two frames mentioned above,
c is the speed of light and
γ is the Lorentz factor (always greater than unity).

2. Length Contraction:

If an object is at rest with respect to a particular reference frame, then the length of the object measured by that frame, L_0 will be the greatest, compared to any other reference frames. In other words, for any reference frames with respect to which, that object is moving at a relative speed v (so is the first frame along with the object), we will find a smaller length, L; where—

$$L = \frac{L_0}{\gamma} = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

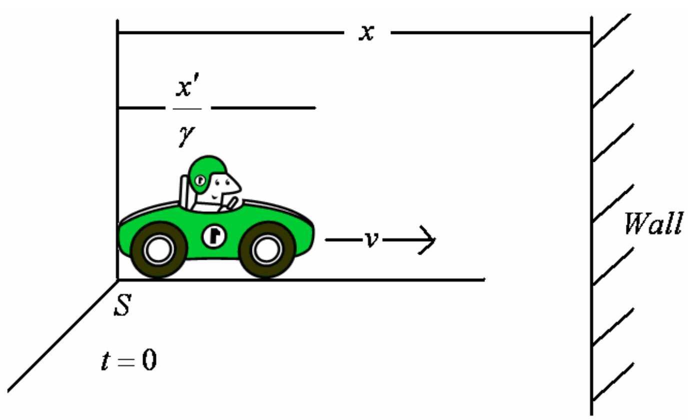
Now, I shall try to bring out the Lorentz Transformations with the help of these two conclusions.

Let's consider two inertial reference frames S and S', having a relative velocity v. When $t=0$ and $t'=0$ for both frames, let their origins be one above another. Let's define this event as EVENT 1.

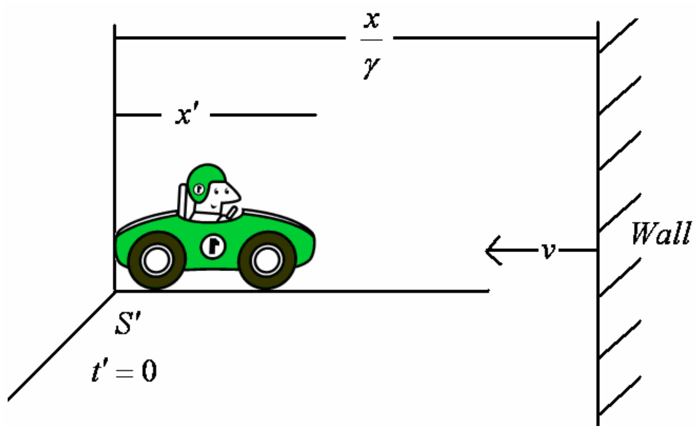
Let again a rigid wall be situated in front of the two frames. Let it be stationary with respect to the S frame and have a distance x from the S origin. Then the other frame S' will observe the wall closing in to its origin at a speed v and therefore will find a smaller distance $\frac{x}{\gamma}$ at $t'=0$.

Again consider a car attached to the origin of S' frame. Since it is stationary with respect to this frame, the length measured, x' will be greatest. And also, since the car is moving at a relative speed, v with respect to the S frame, its therein measured length will be smaller, namely $\frac{x'}{\gamma}$.

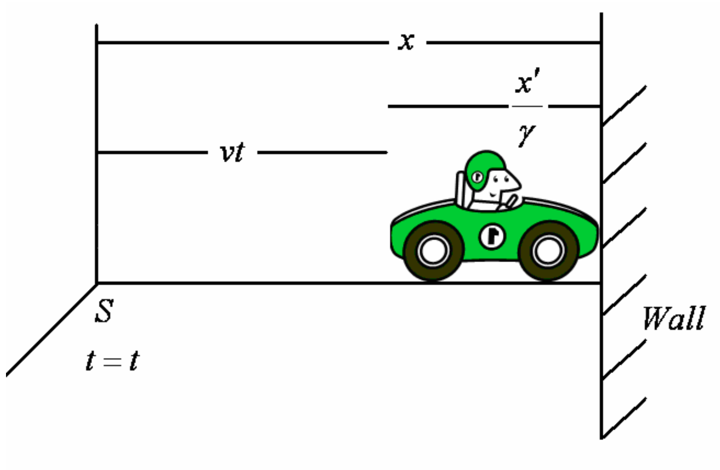
Then at EVENT 1, things will look something like below—



—and—



Now, we define EVENT 2 when the car hits the wall. Let the time co-ordinates for the S frame be then $t = t$. Now, that frame will see the following—

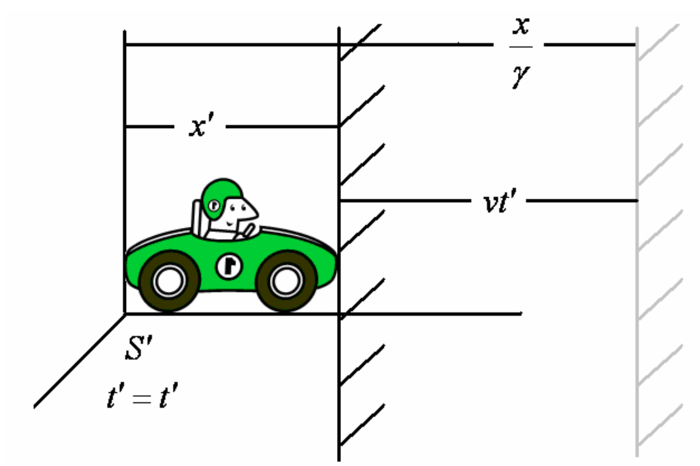


that is,

$$x = vt + \frac{x'}{\gamma}$$
$$\Rightarrow x' = \gamma(x - vt)$$

This is the Spatial Lorenrz Transformation.

Now, for the S' frame; it will see the wall closing in and hitting the stationary car. Let the time co-ordinates be then $t' = t'$. Then—



that is,

$$\frac{x}{\gamma} = vt' + x'$$
$$\Rightarrow x = vt'\gamma + \gamma^2(x - vt) \qquad [\text{putting } x' = \gamma(x - vt)]$$
$$\Rightarrow vt'\gamma = \gamma^2\left(\frac{x}{\gamma^2} - x + vt\right)$$

$$\Rightarrow t' = \gamma\left(\frac{x}{v\gamma^2} - \frac{x}{v} + t\right)$$

$$= \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]$$

$$\therefore t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad [\because 1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}]$$

——which is the Lorentz Transformation for Time.

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[Self generated]

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