Abraham
Math 414 Extra Credit

2.5.7) Let $\{x_n\}$ be a decreasing sequence such that $\sum x_n$ converges.
Show that $\lim_{n\to\infty}(nx_n) = 0$.

Proof: $\sum x_n$ is convergent, therefore it is a cauchy series. By definition, $\exists N \in \mathbb{N}$ such that for any $\epsilon > 0$, and $m > n \geq N$,

$$\left| \sum_{j=1}^{m} x_j - \sum_{j=1}^{n} x_j \right| = \sum_{j=n+1}^{m} x_j < \epsilon \tag{1}$$

Let $m = n + 1$,

$$\sum_{j=n+1}^{n+1} x_j = x_{n+1} < \epsilon$$

Thus, for $n \geq N + 1$, we have $x_n < \epsilon$. Choose N such that for any $\epsilon > 0$, and all $n \geq N + 1$, $x_n < \frac{\epsilon}{n}$

$$\Rightarrow |nx_n - 0| = |nx_n| < n\frac{\epsilon}{n} = \epsilon \tag{2}$$

$$\Rightarrow \lim_{n\to\infty}(nx_n) = 0 \tag{3}$$