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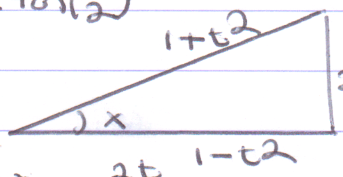
Mathematical functions long assessed problems: 15th November 2014

1. Evaluate the integral $I(y) = \int_0^{\frac{\pi}{2}} \frac{dx}{y+\cos x}$ ← treat y as a constant

$t = \tan(\frac{x}{2})$



Use the substitution $t = \tan(\frac{x}{2})$

on the integral $\int \frac{dx}{y+\cos x}$

$\tan x = \frac{2t}{1-t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2(\frac{x}{2}) = \frac{1}{2}\tan^2(\frac{x}{2}) + \frac{1}{2}$$

$$= \frac{1}{2}t^2 + \frac{1}{2} = \frac{t^2+1}{2}$$

$t = \tan(\frac{x}{2})$

$\frac{dt}{dx} = \frac{1}{2}\sec^2(\frac{x}{2})$

$$\Rightarrow \frac{dx}{dt} = \frac{2}{t^2+1} \Rightarrow dx = \frac{2dt}{t^2+1}$$

$$\int \frac{dx}{y+\cos x} = \int \frac{1}{y+\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{t^2+1} = \int \frac{2dt}{y(t^2+1)+(1-t^2)}$$

$$= \int \frac{2dt}{yt^2+y+1-t^2} = \int \frac{2dt}{(y-1)t^2+(y+1)} = 2\int \frac{dt}{t^2(y-1)+(y+1)}$$

$$= 2\int \frac{1}{(y-1)} \cdot \left(\frac{1}{t^2+\left(\frac{y+1}{y-1}\right)}\right) dt = \frac{2}{y-1}\int \frac{dt}{t^2+\left(\frac{y+1}{y-1}\right)}$$

$a = \sqrt{\frac{y+1}{y-1}}$

use the standard integral of $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$

$$= \frac{2}{y-1} \cdot \sqrt{\frac{y-1}{y+1}} \tan^{-1}\left(\frac{t\sqrt{y-1}}{\sqrt{y+1}}\right) + \psi(y)$$

arbitrary constant when is a function ψ of y so $\psi(y)$

$$= \frac{2}{(y-1)^{1/2}(y+1)^{1/2}} \tan^{-1}\left(\frac{t(y-1)^{1/2}(y-1)^{1/2}}{(y-1)^{1/2}(y+1)^{1/2}}\right) + \psi(y)$$

$$= \frac{2}{\sqrt{y^2-1}} \tan^{-1}\left(\frac{t(y-1)}{\sqrt{y^2-1}}\right) + \psi(y) = \frac{2}{\sqrt{y^2-1}} \tan^{-1}\left(\frac{(y-1)\tan(\frac{x}{2})}{\sqrt{y^2-1}}\right) + \psi(y)$$

$$\text{so } I(y) = \left\{\frac{2}{\sqrt{y^2-1}} \tan^{-1}\left(\frac{(y-1)\tan(\frac{x}{2})}{\sqrt{y^2-1}}\right)\right\}_0^{\frac{\pi}{2}}$$

$$= \left\{\frac{2}{\sqrt{y^2-1}} \tan^{-1}\left(\frac{(y-1)}{\sqrt{y^2-1}}\right)\right\} - \left\{0\right\} = \frac{2}{\sqrt{y^2-1}} \tan^{-1}\left(\frac{(y-1)}{\sqrt{y^2-1}}\right)$$

$\underline{\underline{y > 1}}$