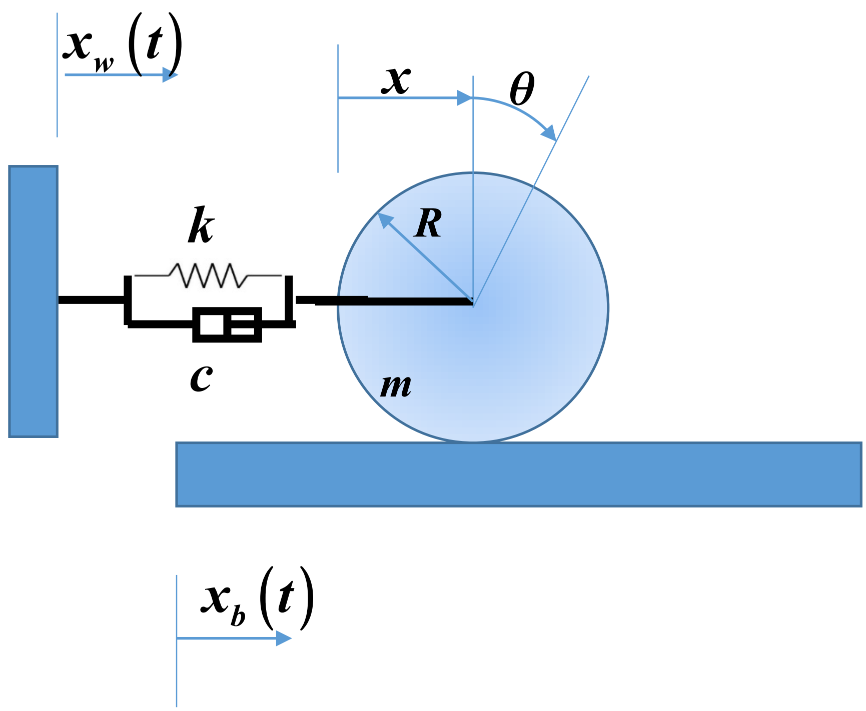
ME357 Problem Set 3

Derive the equation(s) of motion for the systems shown using Newton's Method. For multiple degree of freedom systems put you answer in matrix form. Unless otherwise specified the degrees of freedom are measured from the system's equilibrium position. Unless otherwise stated all motion is small.

1. The wheel is a thin homogeneous disk that rolls without slip.

   The wall moves with a specified motion $x_w(t) = X_w \sin(\omega_w t)$.

   The base moves with a specified motion $x_b(t) = X_b \sin(\omega_b t)$..



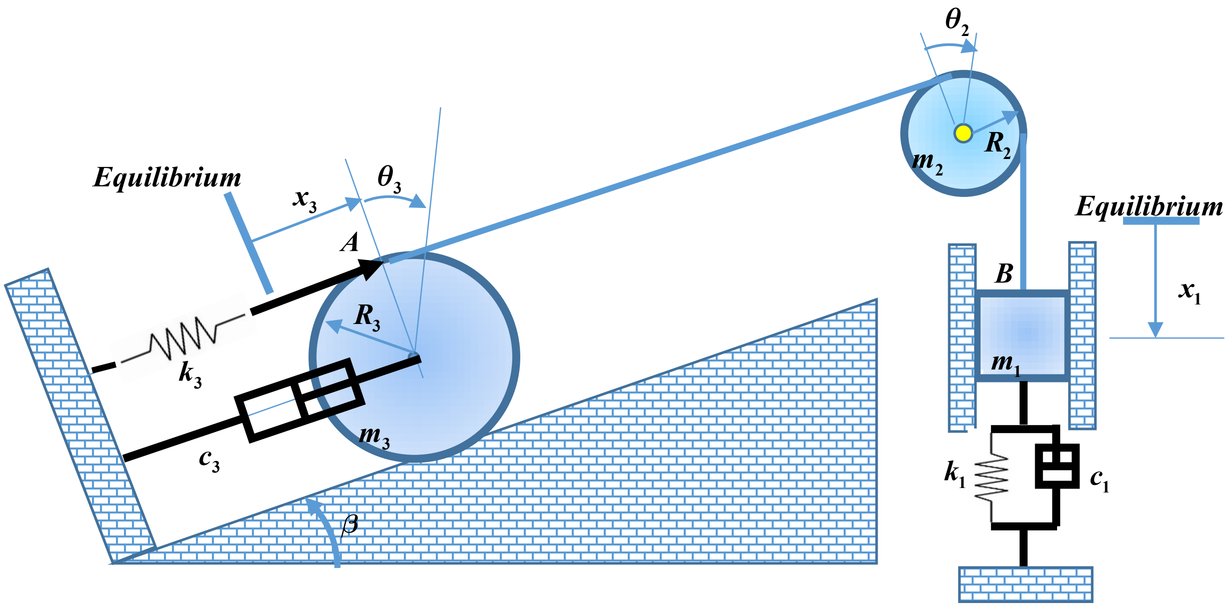
Answer

$$\frac{3}{2}m\ddot{x} + c\dot{x} + kx = -\frac{m}{2}\omega_b^2 X_b \sin(\omega_b t) + c\omega_w X_w \cos(\omega_w t) + kX_w \sin(\omega_w t)$$

2. a. The wheel is a thin uniform disk of mass m_3 that rolls without slip.

   b. The pulley is a thin uniform disk of mass m_2.

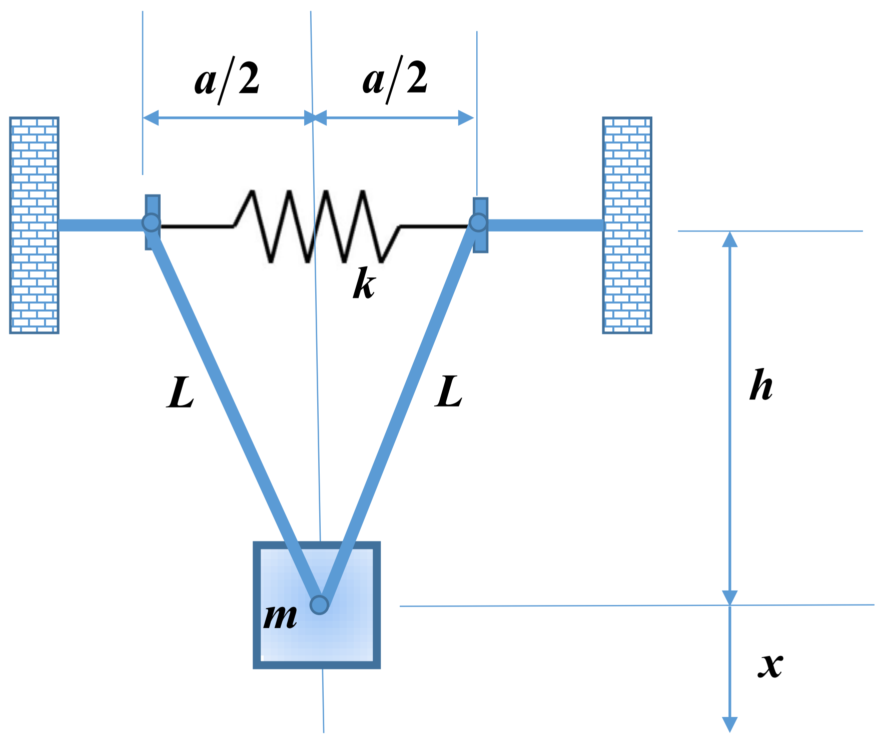
   c. What effect does the inclined plane have? Where would it have an effect?



$$\left(m_1 + \frac{m_2}{2} + \left(\frac{3}{8}\right) m_3\right)\ddot{x} + \left(c_1 + \frac{c_3}{4}\right)\dot{x} + (k_1 + k_3)x = 0$$

3. a. First assume large displacements for x.
   b. Linearize your result from a. for small displacements.

   Assume the two rigid links of length L are massless. The mass m is constrained to move vertically. The system is shown in its equilibrium position.

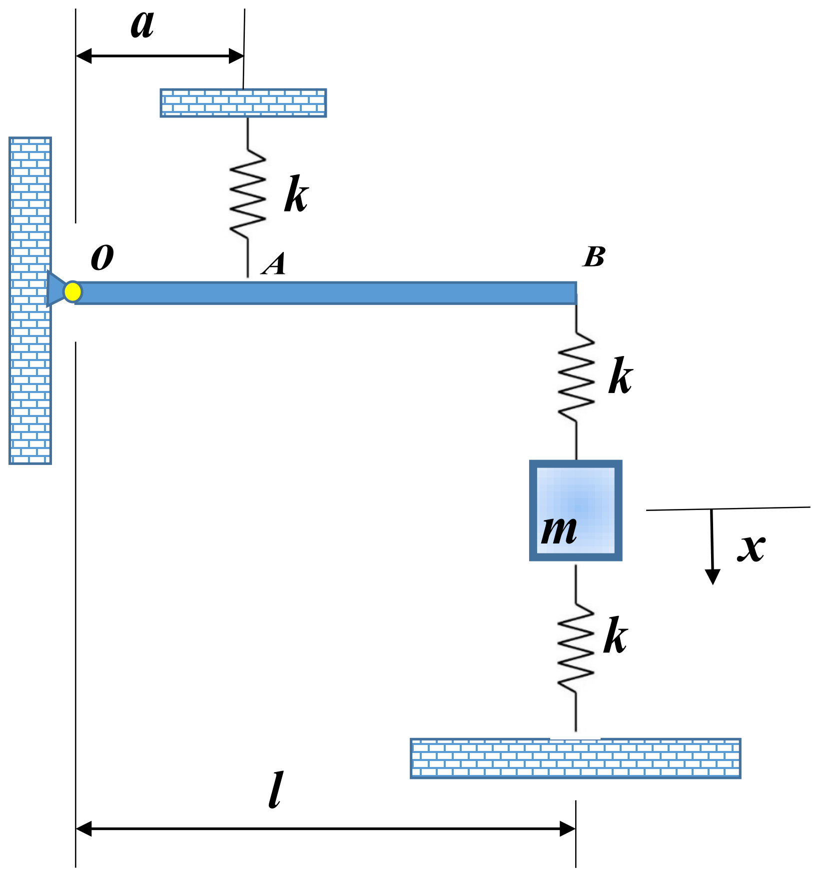


Answer

$$a. \quad m\ddot{x}+\left\{\frac{4k(h+x)\left[1-\sqrt{1-2\left(\frac{2h}{a}\right)\left(\frac{2x}{a}\right)-\left(\frac{2x}{a}\right)^2}\right]}{\sqrt{1-2\left(\frac{2h}{a}\right)\left(\frac{2x}{a}\right)-\left(\frac{2x}{a}\right)^2}}\right\}=0$$

$$b. \quad m\ddot{x}+\left(\frac{4h}{a}\right)^2 x=0$$

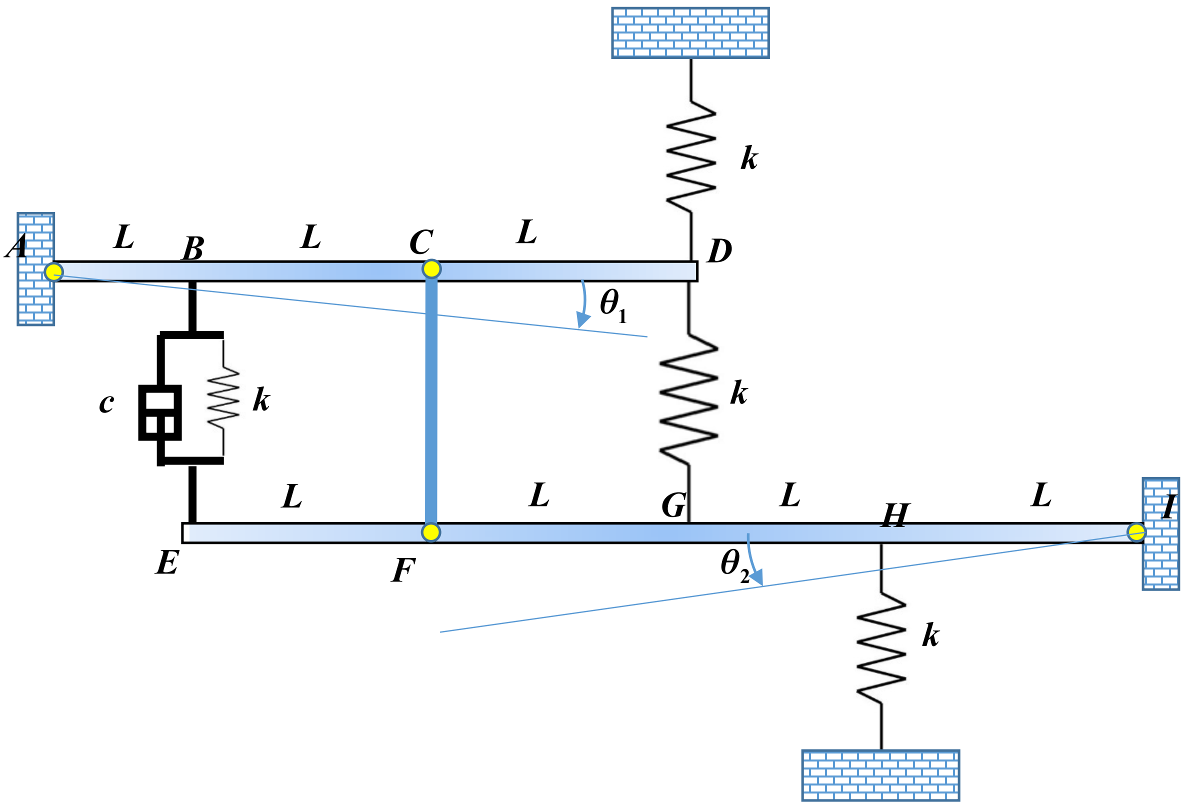
4. Bar OB is rigid and massless.



Answer

$$m\ddot{x}+\left[\frac{1+2\left(\frac{a}{l}\right)^{2}}{1+\left(\frac{a}{l}\right)^{2}}\right]kx=0$$

5. a. Bar AD with mass $\mathbf{3}m$ and EI with mass $\mathbf{4}m$ are rigid, uniform and pinned at A and I.
   b. Bar CF is rigid and massless.
   c. Determine the equation of motion in terms of θ_1 .



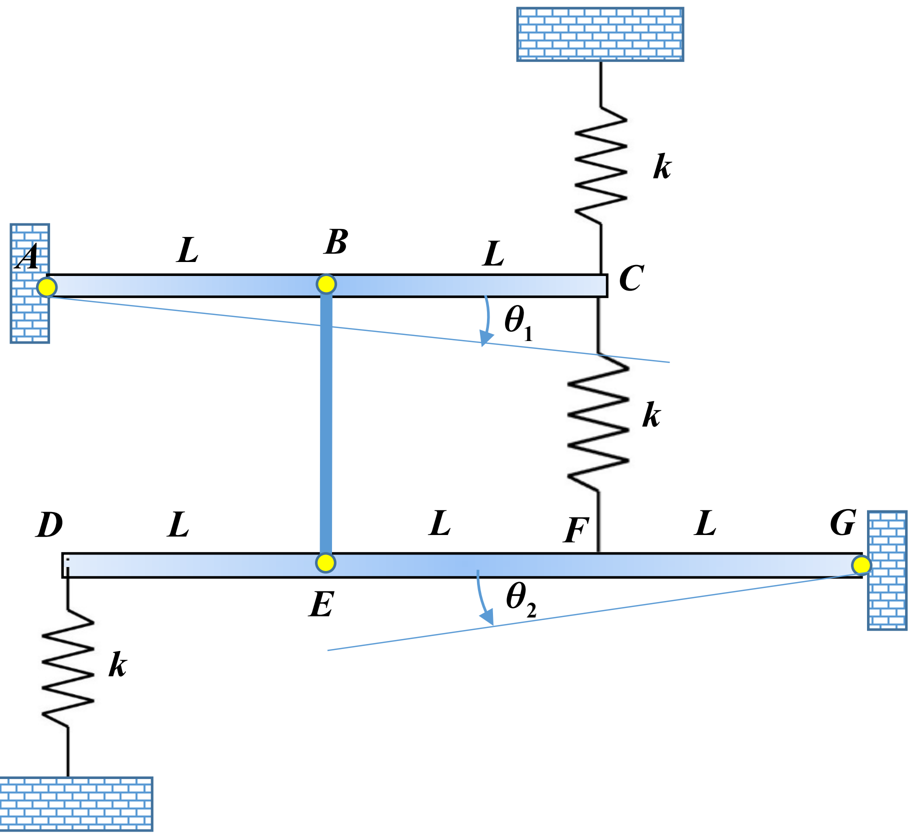
Answer

6. a. Bars AC and DG can be considered as thin rigid uniform bars with masses m and $3m$ respectively.

   b. Bar BE is massless and rigid

   c. Determine the equilibrium position of the two bars as measured from the horizontal.

   d. Determine the equation of motion in terms of θ_1 as measured from the horizontal. What does this assume?



Answer

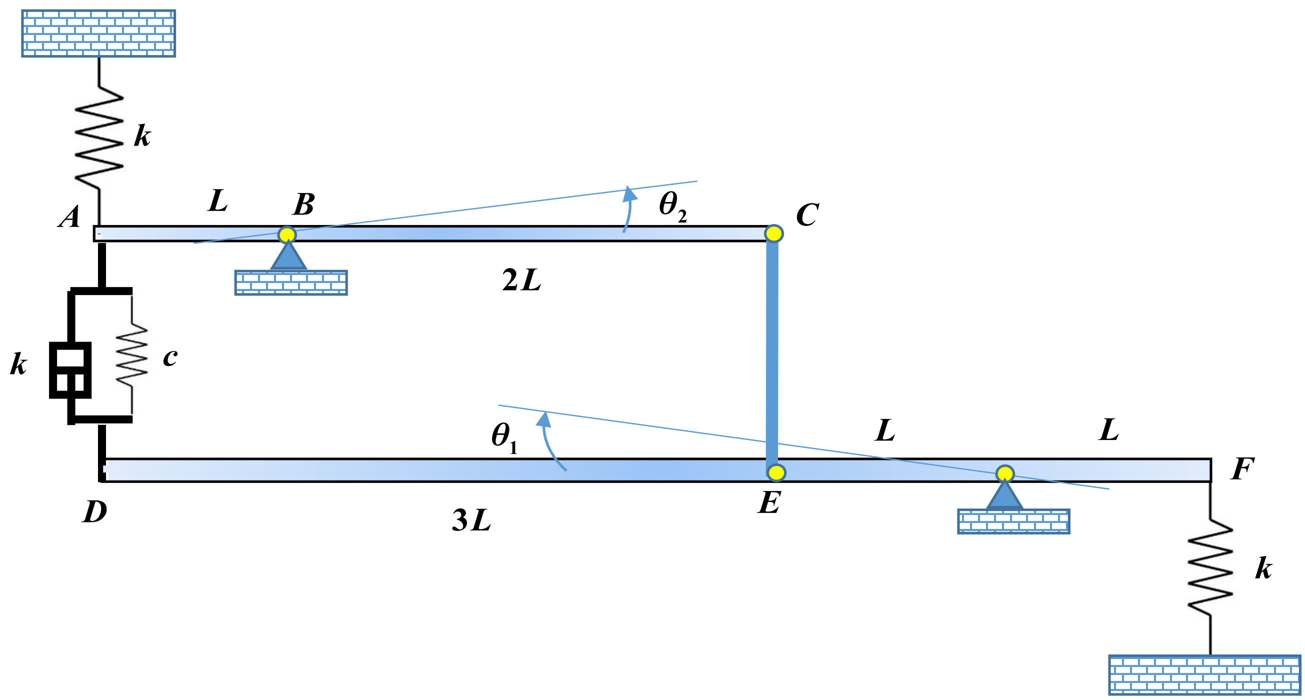
$$a.\ \theta_{1_{eq}} = \frac{13}{34}\left(\frac{mg}{kl}\right)$$

$$b.\ \ddot{\theta}_1 + \frac{102}{48}\left(\frac{k}{m}\right)\theta_1 = 0$$

7. a. Bars AC and DF can be considered as thin rigid uniform bars with masses m_1 and m_2 respectively.

   b. Bar CE is massless and rigid.

   c. Determine the equation of motion in terms of θ_1 .



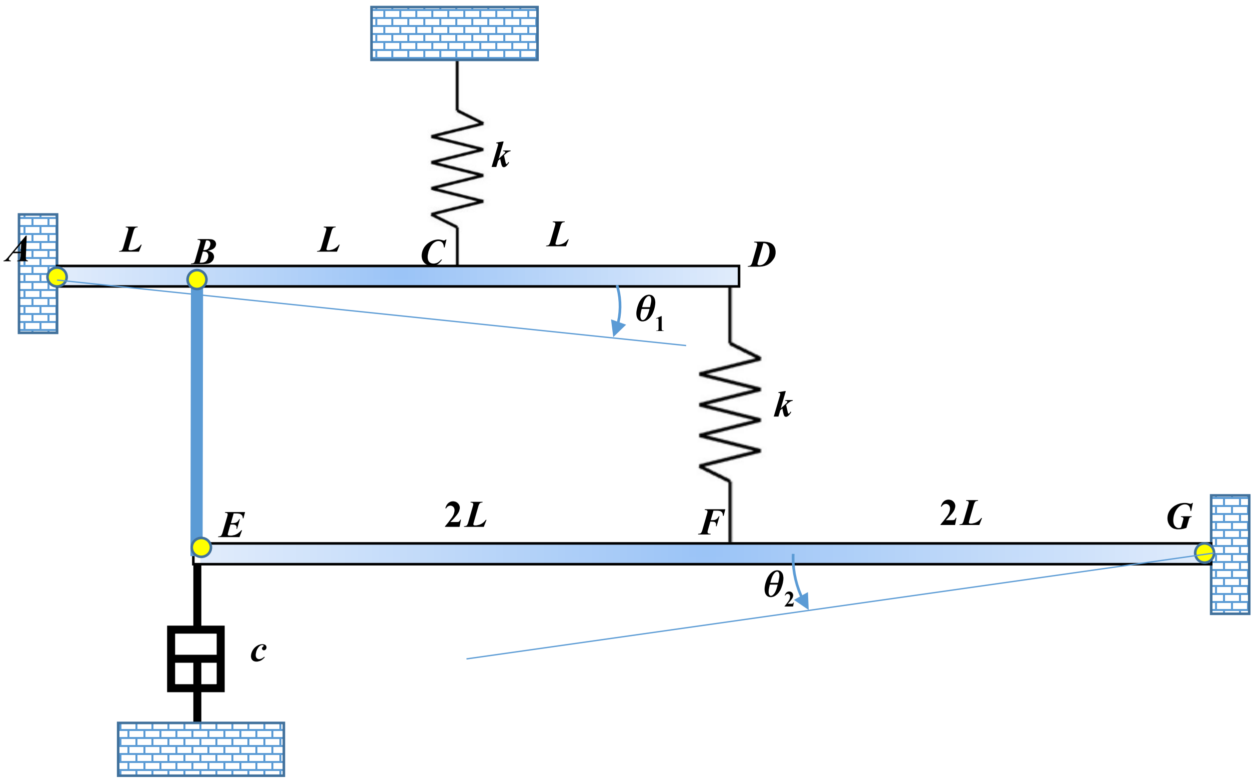
Answer

$$\left(\frac{26}{3}m_1+\frac{1}{2}m_2\right)\ddot{\theta}_1+\left(\frac{81}{2}\right)\dot{\theta}_1+43kL^2\theta_1=0$$

8. a. Bars AD and EG can be considered as thin rigid uniform bars with masses $3m$ and $4m$ respectively.

b. Bar BE is massless and rigid.

c. Determine the equation of motion in terms of θ_1 .



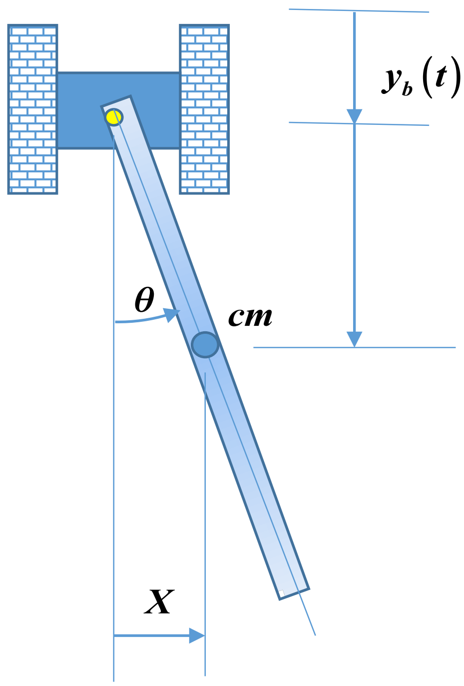
Answer

$$\ddot{\theta}_1 + \left(\frac{3}{31}\right)\left(\frac{c}{m}\right)\dot{\theta}_1 + \left(\frac{123}{124}\right)\left(\frac{k}{m}\right)\theta_1 = 0$$

9. The pendulum is a rigid thin uniform bar.

a. Determine the equation of motion if the base is fixed $y_b(t)=0$

b. Determine the equation if $y_b(t)=Y_b\sin(\omega_b t)$. Where does the vertical motion enter the equation? What does it make gravity/stiffness term behave like?

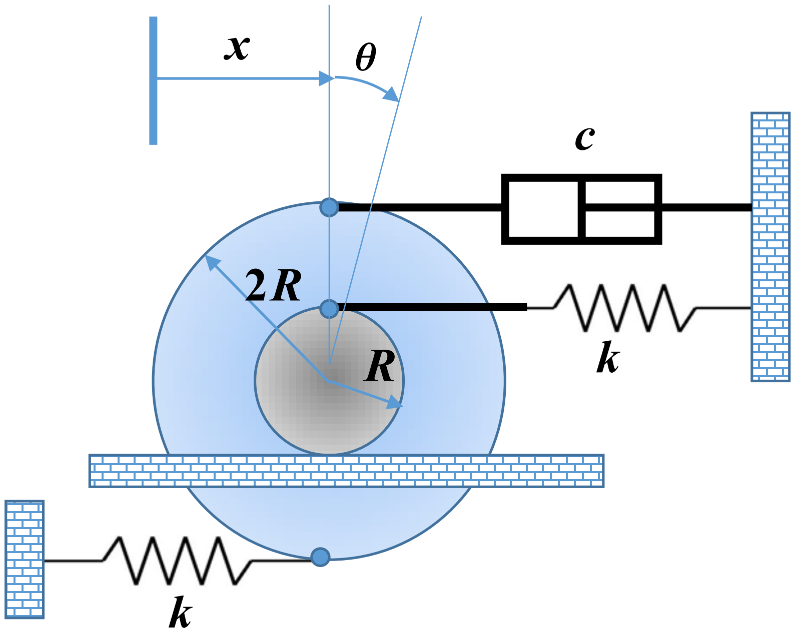


Answer

a. $\ddot{\theta}+\frac{3}{2}\left(\frac{g}{l}\right)\theta=0$

b. $\ddot{\theta}+\frac{3}{2}\left[\frac{g}{l}+\left(\frac{Y_b}{l}\right)\omega_b^2\sin(\omega_b t)\right]\theta=0$

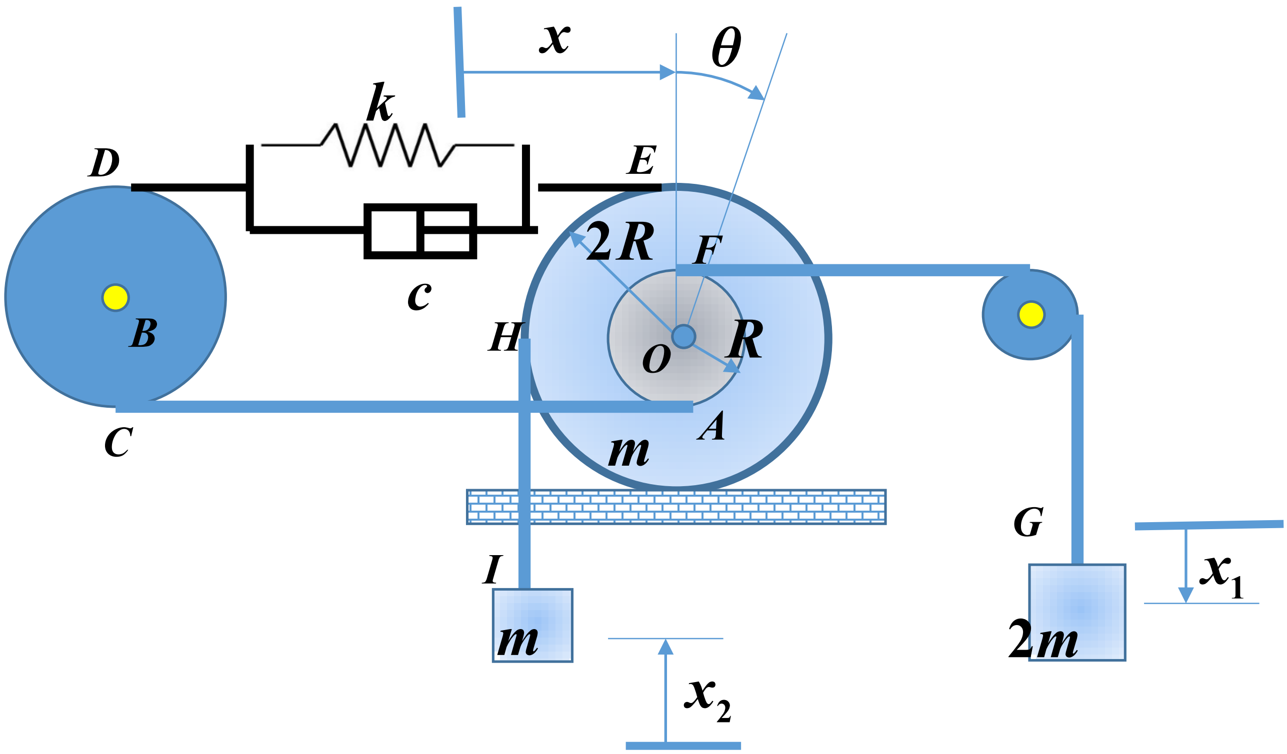
10. The wheel is constructed of 2 uniform thin disks. The inner disk has a radius R and a mass m. and is fixed to the outer disk. The outer disk is a rigid uniform disk with radius $2R$ and mass m. Assume the wheel rolls without slip. Use x as the degree of freedom.



Answer

$$\frac{9}{2}m\ddot{x}+9c\dot{x}+5kx=0$$

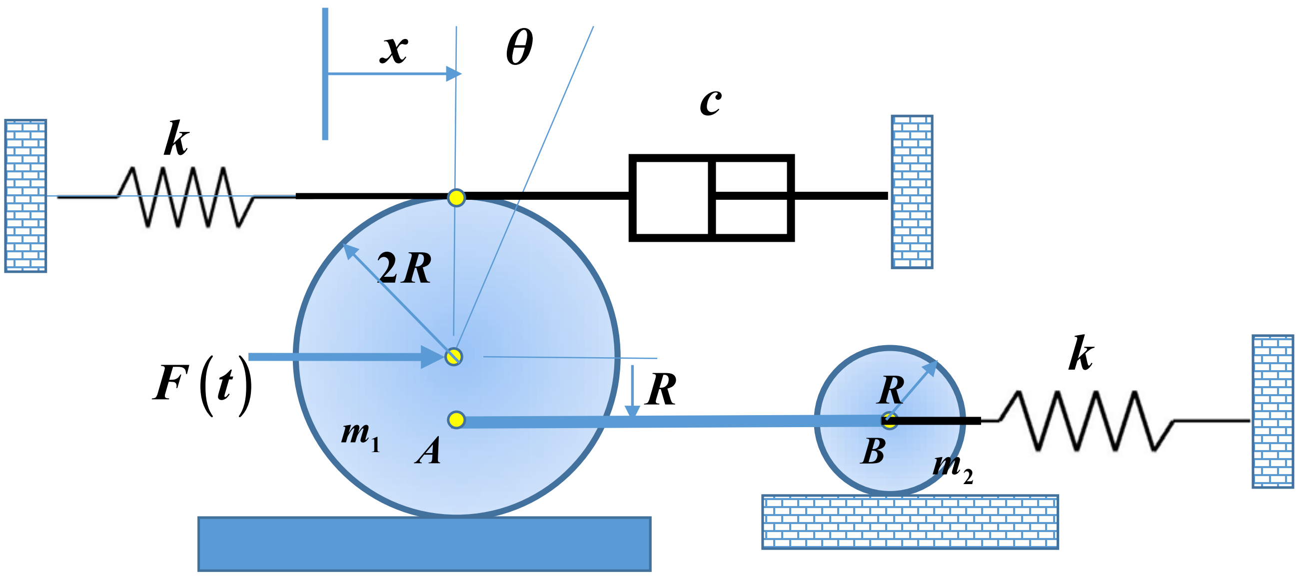
11. Determine the equation of motion in terms of x. Assume the wheel rolls without slip. The two masses are constrained to move only vertically and the pulleys are massless. The wheel has a mass of m and a mass moment of inertia about its center of mass of $4mR^2$. Cord EDCA is elastic and is modeled as a spring/damper. Cords BI and FG are inextensible.



Answer

$$\ddot{x}+\frac{5}{6}\left(\frac{c}{m}\right)\dot{x}+\frac{5}{6}\left(\frac{k}{m}\right)x=0$$

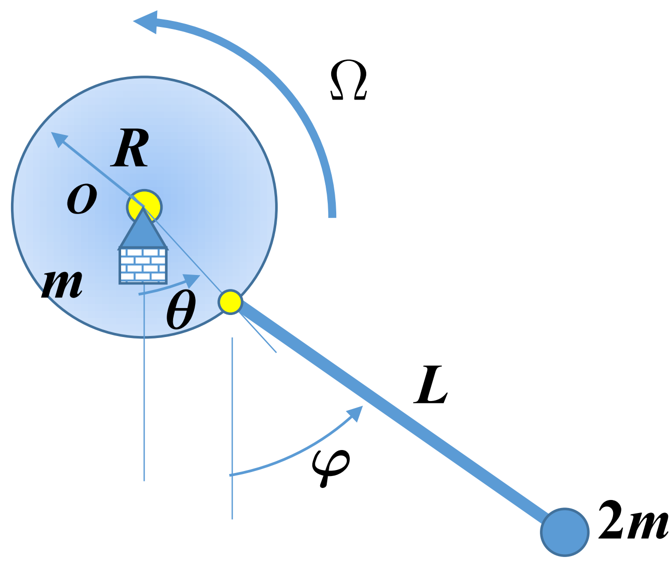
12. The drive wheel is a thin uniform disk of radius $2R$ and mass m_1 The driven disk is radius R and mass m_2 . Both roll without slip. Bar AB is rigid and massless. Determine the equation of motion in terms of x.



Answer :

$$\left[\frac{3}{2}m_1+\frac{3}{8}m_2\right]\ddot{x}+4c\dot{x}+\frac{17}{4}kx=F(t)$$

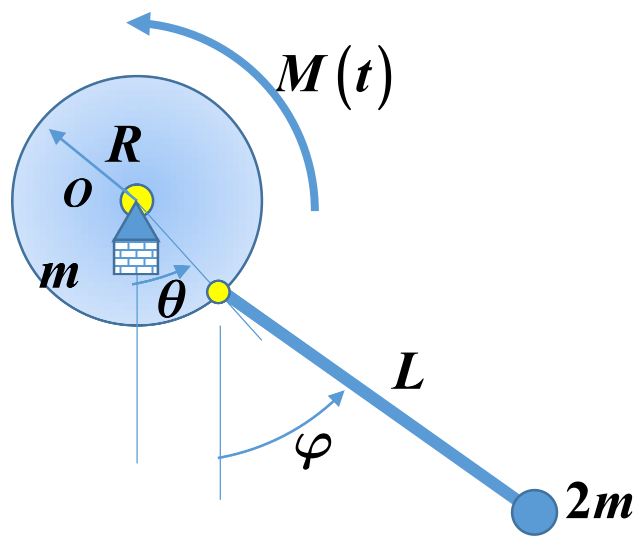
13. Determine the equation of motion in terms of θ. Assume small motion. The pendulum is pinned to the disk which rotates with a constant angular velocity Ω.. The rod is rigid and massless and the mass at its tip is a point mass.



Answer

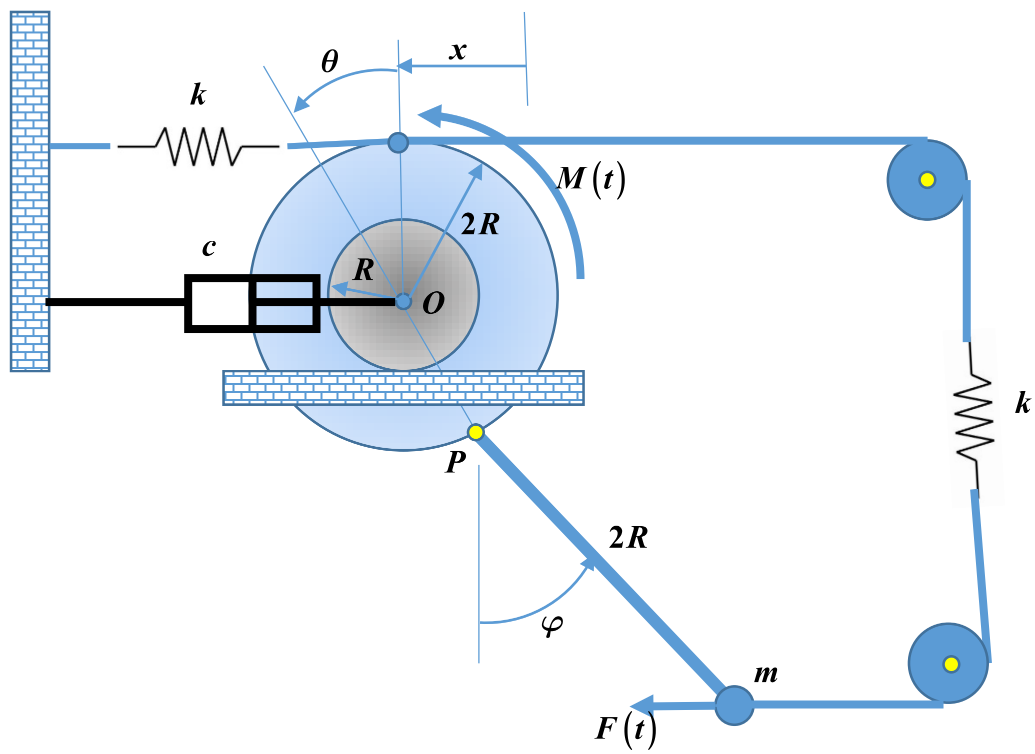
$$\ddot{\varphi}+\left[\frac{g}{L}+\left(\frac{R}{L}\right)\Omega^2\cos(\Omega t)\right]\varphi=\left(\frac{R}{L}\right)\Omega^2\sin(\Omega t)$$

14. The thin homogeneous disk is pinned at O and is driven by a motor that produces a torque, $M(t)$. A point mass of $2m$ is pinned to it with massless bar of length L. Determine the equations of motion in terms of θ and φ.



Answer

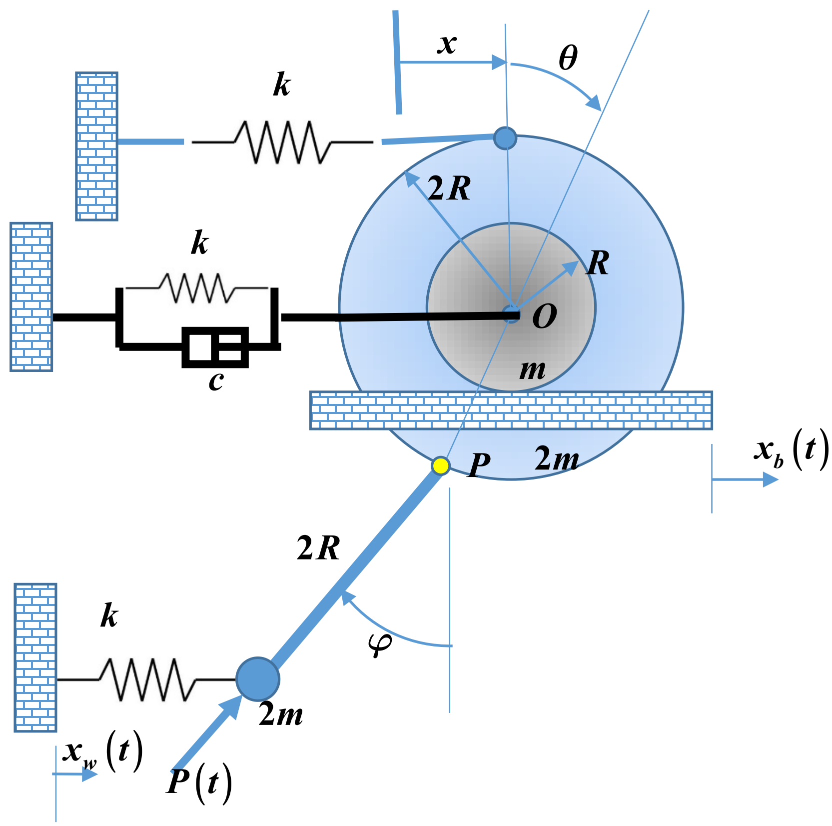
15. Determine the equations of motion in terms of x and φ. Assume small angles and that the wheel rolls without slip. The mass of the thin homogeneous large disk of radius $2R$ is $2m$. The mass of the thin homogeneous inner disk of radius R is m. The rod of length $2R$ is massless and rigid. The two pulleys are massless.



Answer

$$\begin{bmatrix} \frac{11}{2}m & 2mR \\ 2mR & 4mR^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\varphi} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\varphi} \end{Bmatrix} + \begin{bmatrix} 13k + 2\left(\frac{mg}{R}\right) & -4kr \\ -4kr & 4kR^2 + 2mgR \end{bmatrix} \begin{Bmatrix} x \\ \varphi \end{Bmatrix} = \begin{Bmatrix} -F(t) + \frac{M(t)}{R} \\ -2RF(t) \end{Bmatrix}$$

16. Determine the equations of motion for the system shown in terms of x and φ assuming small motion. The block the disk is sitting on moves with a prescribed motion $x_b(t)$. The wall the bottom spring is attached to moves with a prescribed motion $x_w(t)$. The load $P(t)$ is always directed along the pendulum rod. Both disks of radius R and $2R$ are uniform and homogenous with masses m and $2m$ respectively. The wheel rolls without slip.



Answer

$$\begin{bmatrix} \frac{19}{2}m & 4mR \\ 4mR & 8mR^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\varphi} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\varphi} \end{Bmatrix} + \begin{bmatrix} 11k + 4\left(\frac{mg}{R}\right) - \left(\frac{2P}{R}\right) & 2kR + P \\ 2kR & 4kR^2 + 4mgR \end{bmatrix}\begin{Bmatrix} x \\ \varphi \end{Bmatrix} =$$

$$\begin{Bmatrix} \frac{17}{2}m\ddot{x}_b + \left(\frac{4mg}{R}\right)x_b - kx_w - 2\left(\frac{P}{R}\right)x_b \\ 8mR\ddot{x}_b + 4kRx_b - 2kRx_w \end{Bmatrix}$$

17. Repeat Prob 1 if the wheel slips. Use x and θ as your degrees of freedom. Assume a coefficient of friction between the wheel and base of μ.

$$m\ddot{x}+c\dot{x}+\mu mg\left(sign(\dot{x}-\dot{x}_b)\right)+k\left(x-x_b\right)=c\dot{x}_w$$

$$\ddot{\theta}+\left(\frac{2\mu g}{R}\right)\left(sign(\dot{x}-\dot{x}_b)\right)=0$$

18. a. Determine the equations of motion for Example 9 of the notes for large θ.

b. Reduce your results to the equations of motion for small θ.

Answer

$$a.\ (m_1+m_2)\ddot{x}+m_2\left(l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta\right)+c\dot{x}+kx-P\sin\theta=F(t)$$

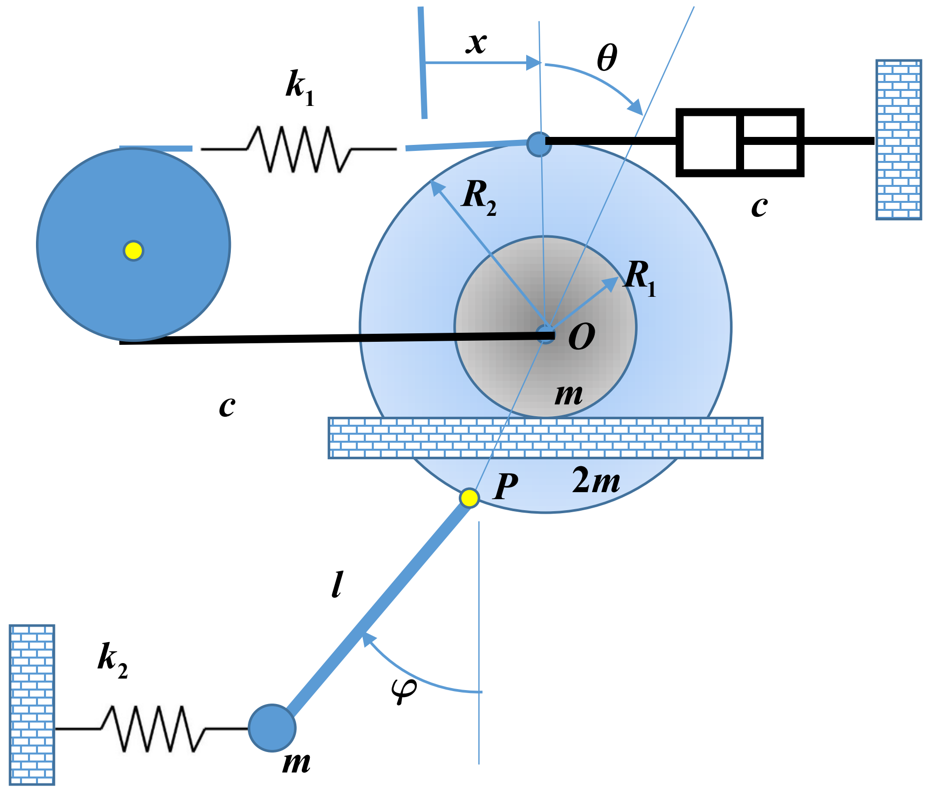
$$m_2\ddot{x}+m_2l\ddot{\theta}+mg\sin\theta=0$$

$$b.\ \begin{bmatrix} m_1+m_2 & m_2l \\ m_2l & m_2l^2 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\theta}\end{Bmatrix}+\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{\theta}\end{Bmatrix}+\begin{bmatrix} k & -P(t) \\ 0 & m_2gl \end{bmatrix}\begin{Bmatrix}x\\ \theta\end{Bmatrix}=\begin{Bmatrix}F(t)\\ 0\end{Bmatrix}$$

19. Complete Example 8 of the notes.

$$\begin{bmatrix} \left(m_1 + \dfrac{m_2}{2}\right) & -\dfrac{m_2}{2} \\ -\dfrac{m_2}{2} & \dfrac{3}{2}m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \left(k_1 + 4k_2 + k_3\right) & -4k_2 \\ -4k_2 & 4k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{M(t)}{R} \\ \dfrac{M(t)}{R} \end{Bmatrix}$$

20. Determine the equations of motion in terms of x and φ. The pulley is massless. The inner homogeneous thin disk of radius R_1 has a mass m. The outer disk of radius R_2 has mass $2m$. The wheel rolls without slip. The pendulum is composed of a massless rod and a point mass.

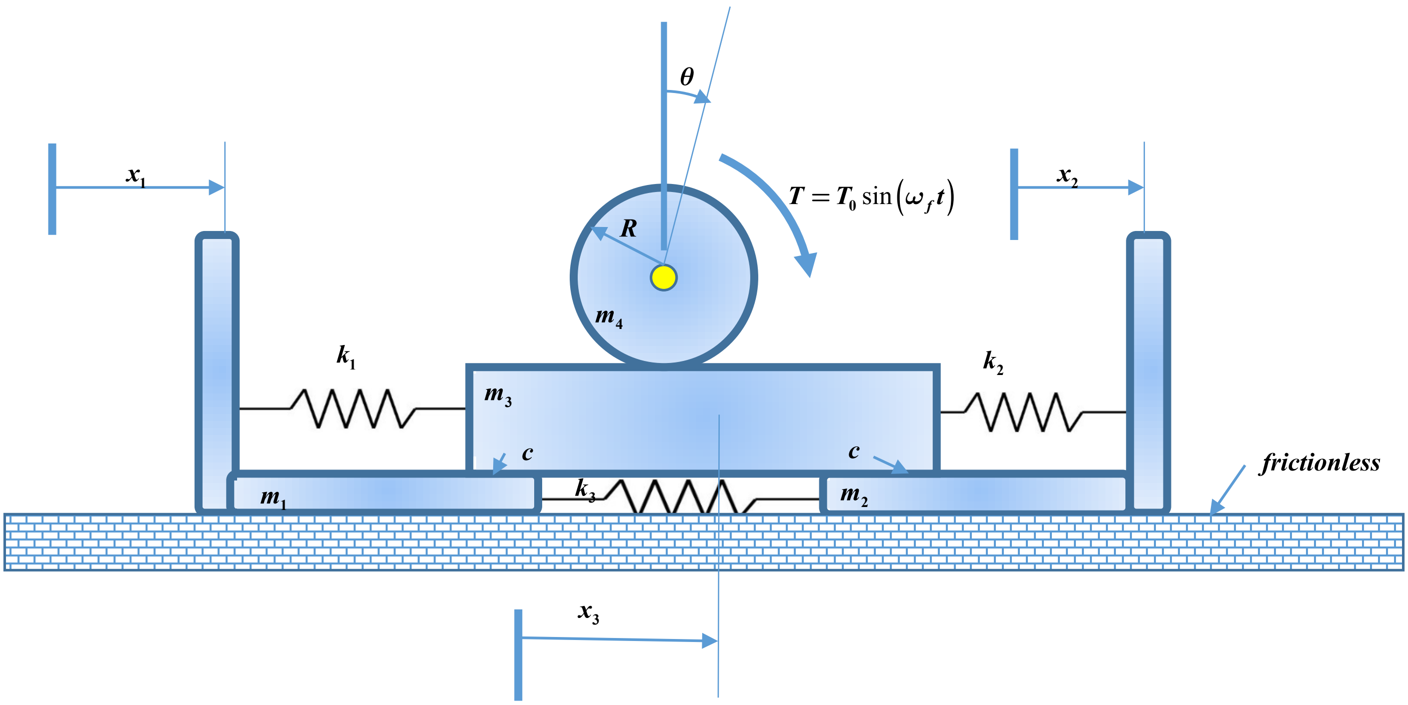


Answer

$$\begin{bmatrix} \left[3m+m\left(1-\left(\frac{R_2}{R_1}\right)\right)^2+\frac{I_{cm}}{R_1^2}\right] & -ml\left(1-\left(\frac{R_2}{R_1}\right)\right) \\ -ml\left(1-\left(\frac{R_2}{R_1}\right)\right) & ml^2 \end{bmatrix}\begin{Bmatrix}\ddot{x}\\ \ddot{\varphi}\end{Bmatrix}+\begin{bmatrix} c\left(1+\left(\frac{R_2}{R_1}\right)\right)^2 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix}\dot{x}\\ \dot{\varphi}\end{Bmatrix}+$$

$$\begin{bmatrix} \left[k_1\left(2+\left(\frac{R_2}{R_1}\right)\right)^2+k_2\left(1-\left(\frac{R_2}{R_1}\right)\right)^2+mg\left(\frac{R_2}{R_1^2}\right)\right] & -k_2 l\left(1-\left(\frac{R_2}{R_1}\right)\right) \\ -k_2 l\left(1-\left(\frac{R_2}{R_1}\right)\right) & k_2 l^2+mgl \end{bmatrix}\begin{Bmatrix}x\\ \varphi\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

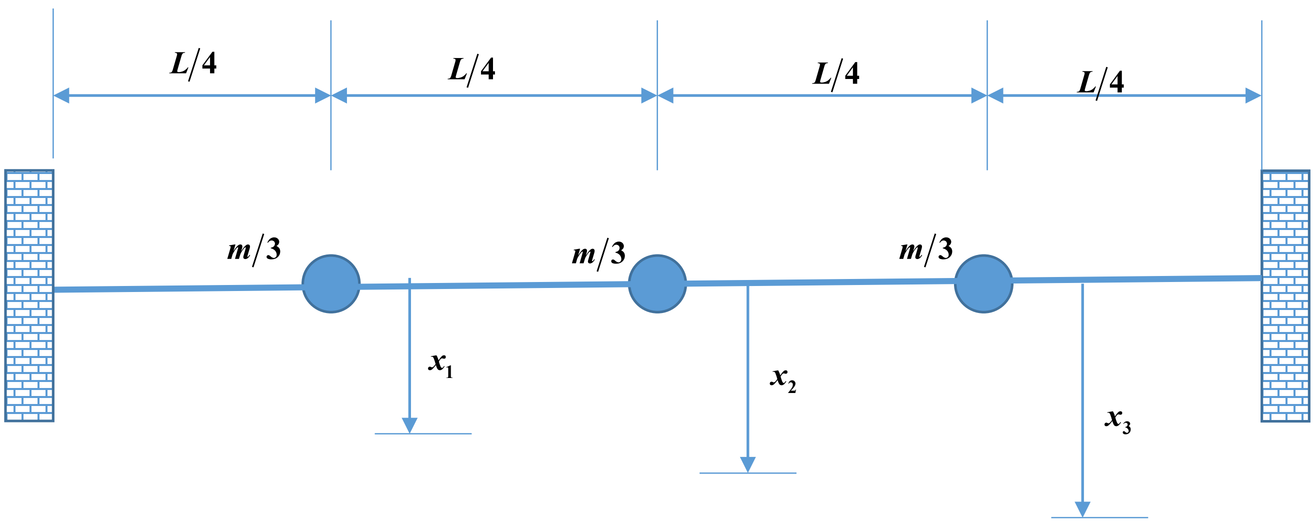
21. Determine the equations of motion in terms of (x_1, x_2, x_3). All $x's$ are measured from the equilibrium position. Masses m_1 and m_2 slide on a frictionless surface. The thin homogeneous disk has a preload on it such that there is no slip, between it and the block m_3. The disk is driven by a torque $T(t)$. There is a viscous interface between m_3 and m_1 and m_3 and m_2. Assume the blocks do not collide or and that m_3 stays in contact with m_1 and m_2.



Answer

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & \left(m_3 + \frac{m_4}{2}\right) \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c & 0 & -c \\ 0 & c & -c \\ -c & -c & -2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 & -k_1 \\ -k_3 & k_2 + k_3 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{T_o}{R} \sin(\omega_r t) \end{Bmatrix}$$

22. The system below models a tightly stretched spring of length L and mass m. The string is stretched with a tension T. Assume the motion is small such that the tension remains constant along the string. Determine the equations of motion in terms of the coordinates x_1, x_2, x_3. By observing the pattern write the equations of motion for 5 masses.



Answer

$$\frac{1}{3}\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + 4\begin{bmatrix} 2\dfrac{T}{L} & -\dfrac{T}{L} & 0 \\ -\dfrac{T}{L} & 2\dfrac{T}{L} & -\dfrac{T}{L} \\ 0 & -\dfrac{T}{L} & 2\dfrac{T}{L} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$