Power Developed by the Science Fair Motor as a Function of Applied Voltage

10/16/06

The instantaneous power expended when the rotor is at an angle θ with respect to the static applied magnetic field is $P(\theta) = V_c(\theta)I(\theta)$, where the EMF developed in the rotor coil as it sweeps through the field is

$$V_c(\theta) = -\frac{d\Phi}{dt} = -\frac{d\Phi}{d\theta}\frac{d\theta}{dt} = \omega\Phi_{max}\sin\theta. \qquad \text{(EQ 1)}$$

Here ω is the angular rotational velocity and the RHS is found from the instantaneous flux in the coil

$$\Phi(\theta) = \Phi_{max}\cos\theta, \qquad \text{(EQ 2)}$$

where Φ_{max} is the flux seen when the coil (solenoid) is parallel to the applied field and intercepting the maximum flux. In general there is an additional contribution $-L(dI/dt)$ to Eq. 1, but at the very low speeds seen in this motor it can be neglected. The current through the winding is determined by the voltage across the coil resistance R

$$I = \frac{V - V_c}{R} \qquad \text{(EQ 3)}$$

where V is the applied battery voltage, so

$$P(\theta) = \frac{(V - V_c)V_c}{R}. \qquad \text{(EQ 4)}$$

The average power expended per each half revolution is the integral of $P(\theta)$ through the angles that voltage is applied (that is through the angles θ_1 to θ_2 defined by the commutator contacts), divided by π,

$$\langle P\rangle = \frac{1}{\pi}\int_{\theta_1}^{\theta_2} P(\theta)d\theta. \qquad \text{(EQ 5)}$$

In the present application, the torque applied to (and by) the motor is constant and is given by the weight mg at the end of the string that hangs from the axle at a radius r_0, $\tau = mgr_0$. In the well known relation between torque and power

$$\langle P \rangle = \omega\tau, \qquad \text{(EQ 6)}$$

τ is constant in this case.

Now we can solve for the motor behavior. Equations 1, 4, 5 and 6 give the angular velocity in terms of applied voltage

$$\omega = \frac{a}{b}V - \frac{\tau}{b}, \qquad \text{(EQ 7)}$$

where the positive constants *a* and *b* are given by

$$a = \frac{\Phi_{max}(\cos\theta_1 - \cos\theta_2)}{\pi R}, \qquad \text{(EQ 8)}$$

$$b = \frac{\Phi_{max}^2}{\pi R}\int_{\theta_1}^{\theta_2}(\sin\theta)^2 d\theta. \qquad \text{(EQ 9)}$$

Substituting Eq. 7 into Eq. 6 gives the final expression for average motor power as a function of voltage,

$$\langle P(V) \rangle = \frac{a\tau}{b}V - \frac{\tau^2}{b}. \qquad \text{(EQ 10)}$$

The power is linear in voltage and intersects the horizontal axis at $V_{min} = \tau/a$, which is the minimum voltage required to start the motor for the given torque $\tau = mgr_0$. We've assumed that friction can be ignored.