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Monash University

SEMESTER TWO EXAMINATIONS 2010

FACULTY OF SCIENCE

EXAM CODE	**MTH3020**
TITLE OF PAPER	**Complex Analysis and Integral Transforms**
EXAM DURATION	Three hours writing time
READING TIME	Ten minutes

THIS PAPER IS FOR STUDENTS STUDYING AT

- ☐ Berwick
- ☑ Clayton
- ☐ Malaysia
- ☐ Distributed Learning
- ☐ Open Learning
- ☐ Caulfield
- ☐ Gippsland
- ☐ Peninsula
- ☐ Enhancement Studies
- ☐ Sth Africa
- ☐ Pharamcy
- ☐ Other

INSTRUCTIONS TO CANDIDATES

1. You may attempt all questions. Each question carries equal marks.
2. You may require the formulæ listed in *Useful Results* at the end of the paper.

AUTHORISED MATERIALS

CALCULATORS	NO
OPEN BOOK	NO
SPECIFICALLY PERMITTED ITEMS	YES

1. *Laplace Transforms* by M R Spiegel

A Complex Analysis

In this Part, all closed simple curves have counterclockwise orientation.

1. (a) Calculate i^{1+i}.

   (b) Determine all the points in the complex plane where the function $f(z) = \tan(z)$ is differentiable and calculate the derivative at those points.

   (c) Determine all points in the region $U = \{ re^{i\theta} \mid 0 < r < \infty \; -\pi < \theta < \pi \} \subset \mathbb{C}$ where the function $f(re^{i\theta}) = \ln(r^2) + i2\theta$ is differentiable and calculate the derivative at those points.

2. (a) Let

   $$\mathcal{C} = \left\{ z(t) = \frac{t\sqrt{2}}{\sqrt{\pi}} e^{it^2} \;\middle|\; 0 \le t \le \sqrt{\frac{\pi}{2}} \right\}$$

   and evaluate

   $$\int_{\mathcal{C}} z e^{z^2} \, dz.$$

   (b) Suppose u is harmonic on the disk $B_R(z_0)$ and that $0 < \rho < R$. Show that

   $$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \, d\theta.$$

   (c) Suppose $c > 0$ and $f(z)$ is an entire function that satisfies $|f(z)| \le c|z|$ for all $z \in \mathbb{C}$. Show that $f(z) = wz$ for some complex constant $w \in \mathbb{C}$.

3. (a) Does there exist a power series of the form $\sum_{j=0}^{\infty} a_j z^j$ that converges at $z = i$ but diverges at $z = 1/2$? (Justify your answer.)

   (b) For $m \in \mathbb{N} = \{ 1, 2, 3, \ldots \}$, define

   $$f_m(z) = \frac{1}{(z - i - 1)^m z}.$$

   i. For each $m \in \mathbb{N}$, find the Laurent series expansion for $f_m(z)$ about $z = i + 1$ and determine the region where the series converges.

   ii. Calculate $\operatorname{res}(f_m(z), z = i + 1)$ for each $m \in \mathbb{N}$.

   iii. Calculate

   $$\int_{|z-i-1|=1} f_m(z) \, dz$$

   for each $m \in \mathbb{N}$.

4. (a) Evaluate

   $$\int_{|z-2|=3/2} \frac{\cosh(z)}{(z-1)^2(z^2-2z)} \, dz.$$

   (b) Use the Residue Theorem to evaluate

   $$\int_0^{\infty} \frac{1}{x^2+4} \, dx.$$

B Integral Transforms

In this Part, list any formulae from Appendices A or B of Spiegel that you use (e.g., [B73]).

5. (a) Using tables, show that $\int_0^\infty J_0(2\sqrt{u\,t}\,)\,J_0(u)\,du = J_0(t)$ by taking the Laplace Transform of the left hand side.

   (b) Use the Convolution Theorem to solve $Y(t) = 2\cos t - \int_0^t (t-u)Y(u)\,du$ for $Y(t)$.

   (c) Use Laplace Transforms to solve $t\,Y'' - (t+2)\,Y = 0$ for $Y(t)$ if $Y(0) = 0$, $Y'(0) = 1$.

6. (a) Show that the inverse Fourier Transform (with respect to λ) of

$$\frac{\sin \lambda t}{\lambda(\lambda^2 - 1)}$$

   is $\frac{1}{4}\,\mathrm{sgn}(t-x)[\cos(t-x) - 1] + \frac{1}{4}\,\mathrm{sgn}(t+x)[\cos(t+x) - 1]$, where $\mathrm{sgn}\,A$ denotes the sign of A (i.e., ± 1). You will require a standard integral given below. Also differentiate the same integral to show that the inverse Fourier Transform of $(\lambda^2 - 1)^{-1}$ is $-\frac{1}{2}\sin x\,\mathrm{sgn}\,x$.

   (b) Use an appropriate Fourier transform to solve

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \delta(x)\sin t\,, \qquad -\infty < x < \infty,\ \ t > 0,$$

   with $U(x,0) = 0$, $U_t(x,0) = 0$, and $U \to 0$ as $x \to \pm\infty$. Hence show that $U(0,t) = \frac{1}{2}(1 - \cos t)$. Here $\delta(x)$ is the Dirac Delta function.

Useful Results

- Cauchy-Riemann equations

$$u_x = v_y \quad u_y = -v_x \qquad \text{(Cartesian form)}$$

$$u_r = \frac{1}{r}v_\theta \quad v_r = -\frac{1}{r}u_\theta \qquad \text{(Polar form)}$$

- Cauchy's Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw \quad n = 0, 1, 2, \ldots$$

- Cauchy's Inequality

  If $f(z)$ is analytic in the disk $B_R(z_0)$, then for any $0 < \rho < R$

$$|f^n(z_0)| \le \frac{n!M_\rho}{\rho^n} \quad n = 0, 1, 2, \ldots$$

  where $M_\rho = \max\big\{|f(z)|\;\big|\;|z - z_0| = \rho\big\}$

- ML Inequality
$$\left|\int_{\mathcal{C}} f(z)\, dz\right| \leq ML$$
where $M = \max_{z\in\mathcal{C}} |f(z)|$ and $L = \ell(\mathcal{C})$

- $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

- $$\int_0^\infty \frac{\sin a\lambda}{\lambda(\lambda^2-1)} d\lambda = \frac{\pi}{2}(\cos a - 1)\,\mathrm{sgn}\, a$$

END OF EXAM