Exercise 21.27. SPHERICAL SHELL OF DUST

Apply the formalism of exercise 21.25 to a collapsing spherical shell of dust [Israel (1967b)]. For the metric inside and outside the shell, take the flat-spacetime and vacuum Schwarzschild expressions (Chapter 23),

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \text{ inside,} \tag{21.176a}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \text{ outside.} \tag{21.176b}$$

Let the "radius" of the shell, as a function of proper time measured on the shell, be

$$R \equiv \frac{1}{2\pi} \times (\text{proper circumference of shell}) = R(\tau). \tag{21.176c}$$

Show that the shell's mass density varies with time as

$$\sigma(\tau) = \mu/4\pi R^2(\tau), \qquad \mu = \text{constant} = \text{"total rest mass"}; \tag{21.176d}$$

and derive and solve the equation of motion

$$M = \mu\left\{1 + \left(\frac{dR}{d\tau}\right)^2\right\}^{1/2} - \frac{\mu}{2R}. \tag{21.176e}$$

Exercise 21.25. EQUATION OF MOTION FOR A SURFACE LAYER

(a) Let $\boldsymbol{u}$ be the "mean 4-velocity" of the matter in a surface layer—so defined that an observer moving with 4-velocity $\boldsymbol{u}$ sees zero energy flux. Let σ be the total mass-energy per unit proper surface area, as measured by such a "comoving observer." Show that the surface stress-energy tensor can be expressed in the form

$$\boldsymbol{S} = \sigma \boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{t}, \text{ where } (\boldsymbol{t} \cdot \boldsymbol{u}) = 0, \tag{21.171}$$

and where $\boldsymbol{t}$ is a symmetric stress tensor.

(b) Show that the component along $\boldsymbol{u}$ of the equation of motion (21.170) is

$$d\sigma/d\tau = -\sigma u^j{}_{|j} + u_j t^{jk}{}_{|k} + u_j[T^{jn}], \tag{21.172}$$

where $d/d\tau = \boldsymbol{u}$. Give a physical interpretation for each term.

(c) Let a_j be that part of the 4-acceleration of the comoving observer which lies in the surface layer Σ. By projecting the equation of motion (21.170) perpendicular to $\boldsymbol{u}$, show that

$$\sigma a_j = -P_{ja}\{t^{ab}{}_{|b} + [T^{an}]\}, \tag{21.173}$$

where P_{ja} is the projection operator

$$P_{ja} = g_{ja} + u_j u_a. \tag{21.174}$$

Give a physical interpretation for each term of equation (21.182).