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$N = 4$ YANG–MILLS THEORY ON THE LIGHT CONE

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The 10-dimensional supersymmetric Yang–Mills theory is constructed in the light-cone gauge. When the theory is dimensionally reduced to four dimensions it is shown that the corresponding $N = 4$ theory is conveniently described in terms of a scalar superfield. This formalism avoids the problem of auxiliary fields but is Lorentz invariant only on the mass shell. Similar formalisms in terms of scalar superfields are also sketched for the other supersymmetric Yang–Mills theories as well as for $N = 8$ supergravity.

1. Introduction

The introduction of supersymmetry into quantum field theories has led to theories with improved quantum properties. In particular the Yang–Mills gauge theory with the maximal supersymmetric extension ($N = 4$) [1, 2] has been found to possess unique properties at the quantum level [3]. There are very strong indications that this theory is a finite quantum field theory. Originally this theory was found in two different ways, one of which was through the construction of supersymmetric Yang–Mills theories in space-times of higher dimension [1]. It was found that the 10-dimensional space is the largest space that carries a supersymmetric Yang–Mills theory and by dimensional reduction this can be taken into 4 dimensions where it emerges as an $N = 4$ extended theory. On the other hand, this theory was already known from the Neveu–Schwarz–Ramond dual model [4], a model which exists only in 10 dimensions. By letting the Regge slope α' tend to zero in the open-string sector of the model one gets the Born terms of a Yang–Mills field coupled to a spinor field, which in fact turns out to be the Born terms of the 10-dimensional supersymmetric Yang–Mills theory [2].

In 4 dimensions the theory contains 1 vector field, 4 spinor fields and 6 scalar fields all in the adjoint representation of the gauge group. The lagrangian is quite complicated and it is difficult to perform extensive quantum calculations in this formalism. Hence much effort has been devoted to the search for a more efficient formalism. The natural candidate for such a formalism is the superspace approach [5], where the supersymmetry is directly built into superfields. The most powerful technique applied so far is the one using $N = 1$ superfield Feynman graphs [6].

However, one generally expects the optimum formalism to be based on superfields which are representations of the full $N=4$ supersymmetry. Such a formalism has turned out to be more difficult to find than was originally anticipated. In fact there are indications [7, 8] that the form in which the theory has been presented so far does not admit a solution to this problem. This fact has made the theory even more elusive and has created an urge for new approaches.

Recently a new development has occurred. The old Neveu–Schwarz–Ramond dual model was known to have inconsistencies such as tachyons. It was conjectured by Gliozzi, Olive and Scherk [2] that the model could be made consistent by projecting out some of the states. This program has now been implemented by Green and Schwarz [9]. They have constructed a new spinning string model where this projection is automatic. This model which is supersymmetric in 10 dimensions still contains the Yang–Mills theory as a certain zero-slope limit. In fact one can now also construct the one-loop contribution to the scattering amplitudes in the same limit [10] and it is found to be remarkably simple, indicating an underlying formalism in terms of a scalar field of some sort.

One key element in the formalism of Green and Schwarz is the use of the light-cone gauge. In fact nobody has found a way to construct the model in a Lorentz covariant fashion. This might be a genuine problem in the sense that such a formalism is overwhelmingly complicated. In this paper we will reconsider the Yang–Mills field theory and show that also for this model the choice of the light-cone gauge is very convenient for writing the theory in a manifestly supersymmetric way. We will show that there is a very compact way to write this theory in terms of a *scalar* superfield. Having given up the manifest Lorentz covariance, however, we will have to check all physical amplitudes explicitly in the final S-matrix to ensure their independence of the Lorentz frame chosen.

An alternative approach to a viable (conventional) superfield formalism is to search for auxiliary fields to close the super-symmetry algebra. It has been clear for some time that if such a set of auxiliary fields exists for the $N=4$ Yang–Mills theory it must be extremely complicated [7, 8]. In the light-cone formalism where we deal only with the physical degrees of freedom, this problem is avoided. The action is invariant under a subalgebra of the supersymmetry algebra and the physical degrees of freedom span a representation of this algebra. Hence in our formulation supersymmetry (or rather a subalgebra of it) is a symmetry of our action while Lorentz invariance is present only on-shell. In the original version of the theory the opposite is true; it is manifestly Lorentz invariant but supersymmetric only on-shell.

The paper is organized as follows: In sect. 2 the 10-dimensional supersymmetric Yang–Mills theory is written in the light-cone gauge and in sect. 3 this theory is dimensionally reduced to 4 dimensions. The corresponding superfield formulation is introduced in sect. 4 and we end in sect. 5 with a brief discussion of light-cone superfield formulations of other supersymmetric theories.

2. The 10-dimensional supersymmetric Yang–Mills theory in the light-cone gauge

Our starting point is the supersymmetric Yang–Mills theory in 10 dimensions. The action is [1]

$$S=\int d^{10}x\left(-\tfrac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a}+\tfrac{1}{2}i\bar{\lambda}^{a}\gamma^{\mu}D_{\mu}\lambda^{a}\right), \tag{2.1}$$

where the vector and spinor fields transform according to the adjoint representation of some semisimple Lie group and the spinor field satisfies both the Majorana and the Weyl constraints (for notations and conventions, see the appendix). We now choose the light-cone gauge:

$$A^{+a}=0\,. \tag{2.2}$$

With this gauge choice it is not necessary to introduce Feynman–Fadde'ev–Popov ghosts and the gauge choice can be directly implemented in the action. We now split the spinor into its two light-cone components:

$$\lambda=\tfrac{1}{2}(\gamma_{+}\gamma_{-}+\gamma_{-}\gamma_{+})\lambda\equiv\lambda_{+}+\lambda_{-}\,. \tag{2.3}$$

Furthermore, we introduce the following non-propagating fields

$$S^{a}=\partial^{+}A^{a}_{+}+\partial^{i}A^{a}_{i}+gf^{abc}\left[\frac{1}{\partial^{+}}(A^{b}_{i}\partial^{+}A^{c}_{i})\right.$$
$$\left.+\tfrac{1}{2}i\frac{1}{\partial^{+}}(\bar{\lambda}^{b}_{+}\gamma^{+}\lambda^{c}_{+})\right], \tag{2.4}$$

$$\chi^{a}_{-}=\lambda^{a}_{-}+\frac{1}{2\partial^{+}}\gamma_{-}\gamma^{\mu}\partial_{\mu}\lambda^{a}_{+}$$
$$-\tfrac{1}{2}gf^{abc}\frac{1}{\partial^{+}}(\gamma_{-}\gamma^{i}\lambda^{c}_{+}A^{ib})\,. \tag{2.5}$$

Eventually in the effective action we will Fourier transform to momentum space. When this is done the factors $(\partial^{+})^{-1}$ will be harmless. Note, however, that some care must be exercised when they are used to perform "partial integrations". We now write the full action as

$$S=\int d^{10}x\left\{\tfrac{1}{2}A^{a}_{i}\Box A^{a}_{i}-\tfrac{1}{4}i\bar{\lambda}_{+}\gamma_{-}\frac{\Box}{\partial^{+}}\lambda_{+}\right.$$
$$+gf^{abc}\left[\partial_{i}A^{a}_{i}\frac{1}{\partial^{+}}(\partial^{+}A^{b}_{j}A^{c}_{j})\right.$$
$$-\partial_{i}A^{a}_{j}A^{b}_{i}A^{c}_{j}$$
$$+\tfrac{1}{2}i\bar{\lambda}^{a}_{+}A^{b}_{i}\gamma_{i}\gamma_{j}\gamma_{-}\frac{\partial_{j}}{\partial^{+}}\lambda^{c}_{+}$$

$$
\begin{aligned}
&+\tfrac{1}{2}i\,\partial_i A_i^a \frac{1}{\partial^+}(\bar\lambda_+^b \gamma_- \lambda_+^c)\Big]\\
&+g^2 f^{abc} f^{ade}\Big[-\tfrac{1}{4}A_i^b A_j^c A_i^d A_j^e\\
&-\frac{1}{2}\frac{1}{\partial^+}(\partial^+ A_i^b A_i^c)\frac{1}{\partial^+}(\partial^+ A_j^d A_j^e)\\
&-\tfrac{1}{4}i\frac{1}{\partial^+}(\bar\lambda_+^b \gamma_i A_i^c)\gamma_j A_j^d \gamma_- \lambda_+^e\\
&-\tfrac{1}{2}i\frac{1}{\partial^+}(\partial^+ A_j^b A_j^c)\frac{1}{\partial^+}(\bar\lambda_+^d \gamma_- \lambda_+^e)\\
&+\frac{1}{8}\frac{1}{\partial^+}(\bar\lambda_+^b \gamma_- \lambda_+^c)\frac{1}{\partial^+}(\bar\lambda_+^d \gamma_- \lambda_+^e)\Big]\\
&+\tfrac{1}{2}S^a S^a + \tfrac{1}{2}i\bar\chi_-^a \gamma_+ \partial^+ \chi_-^a\Big\}\,.
\end{aligned}
\tag{2.6}
$$

In a functional integral we can change integration variables from A_+^a and λ_-^a to S^a and χ_-^a and integrate out the terms depending on these auxiliary fields. This integration should be performed with care so that possible topological effects are not eliminated. However, we will use this formalism mainly for perturbation expansions and for this purpose we can neglect new functional dependences introduced by topology. Therefore, from now on, we simply drop the terms depending on S^a and χ_-^a. Note that it is at this stage we lose Lorentz invariance. However, it is possible to represent the Lorentz transformations non-linearly on A^i and A^- and on λ_+ and λ_- by making compensating gauge transformations. Since under those transformations S^a and χ_-^a do transform it is clear that we lose Lorentz invariance off-shell if we eliminate them.

The original action (2.1) is invariant under the supersymmetry transformations

$$\delta A_\mu^a = i\bar\alpha\gamma_\mu\lambda^a\,, \tag{2.7}$$

$$\delta\lambda^a = -\tfrac{1}{2}\gamma^{\mu\nu}\alpha F_{\mu\nu}^a\,. \tag{2.8}$$

The, by now classical, problem associated with these transformations, is that they do not close to form an algebra. To achieve closure it is necessary to introduce auxiliary fields but, alas, it is not known how to do this. However, we now restrict the parameter α of the supersymmetry transformations to α_- [cf. (2.3)]. Then the physical degrees of freedom transform as

$$\delta A_i^a = i\bar\alpha\gamma_i\lambda_+^a\,, \tag{2.9}$$

$$\delta\lambda_+^a = \gamma^i\gamma_+\alpha\,\partial^+ A_i^a\,. \tag{2.10}$$

This algebra closes. It is a subalgebra of the supersymmetry algebra. α_- and λ_+ are both 8-dimensional spinors under SO(8) implying that the action has an invariance corresponding to 8 grassmannian generators, which in fact is the largest symmetry one can expect.

We can also easily understand the symmetry algebraically. Consider the 10-dimensional supersymmetry algebra

$$\{Q_\alpha, \bar{Q}^\beta\} = (\gamma^\mu)_\alpha{}^\beta P_\mu\,, \tag{2.11}$$

where $(\gamma^\mu)_\alpha{}^\beta$ contains a Weyl projection. If we project with $\frac{1}{2}\gamma_+\gamma_-$ we obtain

$$\{Q_{+\alpha}, \bar{Q}^\beta_+\} = (\gamma_+)_\alpha{}^\beta P^+\,, \tag{2.12}$$

where Q_+ transforms as a spinor under SO(8). This algebra, which we will call restricted supersymmetry, is essentially just a Clifford algebra. Under a transformation the parameter must be "minus-projected", i.e. α_-.

At this stage one can introduce a superfield with an anti-commuting coordinate θ which is an 8-dimensional spinor. We will, however, delay the presentation of that formalism to a future publication and concentrate here on the form of the theory obtained by dimensional reduction to the 4-dimensional Minkowski space.

3. Dimensional reduction to 4 dimensions

In the usual dimensional reduction [11] one imagines a certain number of space dimensions to be circular. Then in the simplest case we can demand the fields not to depend on the corresponding coordinates so that those can be integrated away in the action. In the case we investigate we choose 6 of the 8 transverse coordinates to be circular. We then break the SO(8) invariance

$$\begin{aligned} \mathrm{SO}(8) &\to \mathrm{SO}(6)\times\mathrm{SO}(2) \\ &\approx \mathrm{SU}(4)\times\mathrm{U}(1)\,. \end{aligned} \tag{3.1}$$

In this way we obtain a theory with an SU(4) internal global symmetry. The SO(2) part will be just the helicity group. Consider so the action (2.6) under the dimensional reduction above. For the Bose fields we make the following identifications $(m, n, \ldots = 1, 2, 3, 4)$:

$$A^a \equiv \sqrt{\tfrac{1}{2}}(A^a_1 + iA^a_2)\,, \tag{3.2}$$

$$C^{m4a} \equiv \sqrt{\tfrac{1}{2}}(A^a_{m+3} + iA^a_{m+6})\,, \qquad (m \neq 4)\,, \tag{3.3}$$

$$\bar{C}^a_{mn} = \tfrac{1}{2}\epsilon_{mnpq}C^{pqa} = (\overline{C^{mna}})\,. \tag{3.4}$$

To reduce the spinor terms we choose a particular representation of the γ-matrices in 10 dimensions:

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \otimes I_8\,, \tag{3.5}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \otimes I_8\,, \qquad i = 1, 2, 3\,, \tag{3.6}$$

$$\gamma_{mn} = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \otimes \begin{pmatrix} 0 & \rho^{mn} \\ \rho_{mn} & 0 \end{pmatrix}, \qquad m, n = 1, 2, 3, 4\,, \tag{3.7}$$

where we use the matrices γ_{mn} (antisymmetric in m and n) instead of $\gamma_4, \gamma_5, \dots, \gamma_9$. ρ^{mn} and ρ_{mn} are 4×4 matrices:

$$(\rho^{mn})_{kl} = \delta_{mk}\delta_{nl} - \delta_{nk}\delta_{ml}\,, \tag{3.8}$$

$$(\rho_{mn})_{kl} = \tfrac{1}{2}\epsilon_{mnpq}(\rho^{pq})_{kl} = \epsilon_{mnkl}\,. \tag{3.9}$$

In this representation

$$\gamma^{11} = \gamma^0\gamma^1 \cdots \gamma^9 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \otimes \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}, \tag{3.10}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}. \tag{3.11}$$

If we now demand the 32-component spinor λ to be anti-Weyl, Majorana (see the appendix) and "+"-projected we end up with a spinor

$$\lambda_+ = \begin{pmatrix} 0 \\ 0 \\ \chi^1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \chi^4 \\ 0 \\ 0 \\ \bar{\chi}_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\chi}_4 \\ 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

Note that χ^m is a complex Grassmann number. Finally we have arrived at a formalism where the spinors carry no space-time indices.

We can now introduce (3.2)–(3.12) into the action and obtain the $N=4$ light-cone action

$$
\begin{aligned}
S = \int \mathrm{d}^4x \Big\{ & \bar{A}^a \Box A^a + \tfrac{1}{4}\bar{C}^a_{mn} \Box C^{mna} \\
& + \sqrt{\tfrac{1}{2}} i \bar{\chi}^a_m \frac{\Box}{\partial^+} \chi^{ma} \\
& + g f^{abc} \Big[-2 \frac{\bar{\partial}}{\partial^+} A^a \partial^+ \bar{A}^b A^c \\
& - \frac{1}{2} \frac{\bar{\partial}}{\partial^+} A^a \partial^+ \bar{C}^b_{mn} C^{mnc} + \tfrac{1}{2} A^a \bar{\partial} \bar{C}^b_{mn} C^{mnc} \\
& + i\sqrt{2} \frac{\bar{\partial}}{\partial^+} A^a \bar{\chi}^b_m \chi^{mc} - i\sqrt{2} A^a \chi^{mb} \frac{\bar{\partial}}{\partial^+} \bar{\chi}^c_m \\
& + i\sqrt{2} \frac{\bar{\partial}}{\partial^+} \bar{\chi}^a_m \bar{\chi}^b_n C^{mnc} + \text{complex conjugate} \Big] \\
& + g^2 f^{abc} f^{ade} \Big[-2 \frac{1}{\partial^+} (\partial^+ A^b \bar{A}^c) \frac{1}{\partial^+} (\partial^+ \bar{A}^d A^e) \\
& - \tfrac{1}{2} C^{mnb} A^c \bar{C}^d_{mn} \bar{A}^e - \frac{1}{2} \frac{1}{\partial^+} (\partial^+ \bar{A}^b A^c + \partial^+ A^b \bar{A}^c) \frac{1}{\partial^+} (\partial^+ \bar{C}^d_{mn} C^{mne}) \\
& - \tfrac{1}{16} C^{mnb} C^{pqc} \bar{C}^d_{mn} \bar{C}^e_{pq} - \frac{1}{8} \frac{1}{\partial^+} (\partial^+ \bar{C}^b_{mn} C^{mnc}) \frac{1}{\partial^+} (\partial^+ \bar{C}^d_{pq} C^{pqe}) \\
& - i\sqrt{2} \frac{1}{\partial^+} (\bar{\chi}^b_m \bar{A}^c) A^d \chi^{me} \\
& + i\sqrt{2} \frac{1}{\partial^+} (\chi^{mb} A^c) \bar{C}^d_{mn} \chi^{ne} \\
& + i\sqrt{2} \frac{1}{\partial^+} (\bar{\chi}^b_m \bar{A}^c) C^{mnd} \bar{\chi}^e_n \\
& + i\sqrt{2} \frac{1}{\partial^+} (\bar{\chi}^b_m C^{mnc}) \bar{C}^d_{np} \chi^{pe} \\
& + i\sqrt{2} \frac{1}{\partial^+} (\partial^+ A^b \bar{A}^c + \partial^+ \bar{A}^b A^c + \tfrac{1}{2}\partial^+ \bar{C}^b_{mn} C^{mnc}) \frac{1}{\partial^+} (\bar{\chi}^d_p \chi^{ep}) \\
& + \frac{1}{\partial^+} (\bar{\chi}^b_m \chi^{mc}) \frac{1}{\partial^+} (\bar{\chi}^d_n \chi^{ne}) \Big] \Big\} .
\end{aligned}
\tag{3.13}
$$

The derivative ∂ (and its complex conjugate $\bar{\partial}$) is defined by

$$\partial \equiv \sqrt{\tfrac{1}{2}}(\partial_1 + i\partial_2)\,. \tag{3.14}$$

The SU(4) invariance of the action is now obvious. The action (3.13) could, of course, also have been obtained directly from the 4-dimensional action of ref. [1]. The transformations (2.9) and (2.10) now take the form

$$\delta A^a = i\alpha^m \bar{\chi}^a_m\,, \tag{3.15}$$

$$\delta C^{mna} = -i(\alpha^m \chi^{na} - \alpha^n \chi^{ma} + \epsilon^{mnpq}\bar{\alpha}_p \bar{\chi}^a_q)\,, \tag{3.16}$$

$$\delta \chi^{ma} = \sqrt{2}\alpha^m \partial^+ \bar{A}^a + \sqrt{2}\bar{\alpha}_n \partial^+ C^{mna}\,. \tag{3.17}$$

4. The light-cone superfield formulation

Having the action (3.13), invariant under the restricted supersymmetry (3.15)–(3.17), it is now natural to attempt a superfield formulation. In order to do so we start by considering the algebra (2.12). Utilizing the γ-matrix representation (3.5)–(3.11) and the fact that Q_+ has the opposite handedness to λ_+, we can write Q_+ as

$$Q_+ = \begin{pmatrix} 0 \\ q^1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \bar{q}_1 \\ 0 \\ \vdots \end{pmatrix}, \tag{4.1}$$

and the algebra (2.12) reduces to

$$\{q^m, \bar{q}_n\} = -\sqrt{2}\delta^m_n p^+\,, \tag{4.2}$$

exhibiting a manifest SU(4) covariance.

As is conventionally done we represent this algebra on a Grassmann parameter θ^m with its complex conjugate $\bar{\theta}_m$ as follows:

$$q^m = -\frac{\partial}{\partial\bar{\theta}_m} + \sqrt{\tfrac{1}{2}}i\theta^m\partial^+\,, \tag{4.3}$$

$$\bar{q}_n = \frac{\partial}{\partial\theta^n} - \sqrt{\tfrac{1}{2}}i\bar{\theta}_n\partial^+\,. \tag{4.4}$$

A general superfield will now be a function of θ^m and $\bar{\theta}_n$. However, such a superfield will not be an irreducible representation of the algebra (4.2). In fact we can also construct covariant derivatives d^m and $\bar{\mathrm{d}}_n$

$$\mathrm{d}^m = -\frac{\partial}{\partial\bar{\theta}_m} - \sqrt{\tfrac{1}{2}}i\theta^m\partial^+ \,, \tag{4.5}$$

$$\bar{\mathrm{d}}_n = \frac{\partial}{\partial\theta^n} + \sqrt{\tfrac{1}{2}}i\bar{\theta}_n\partial^+ \,, \tag{4.6}$$

which anticommute with q^m and $\bar{q}_n$.

This means that we can impose a "chirality" condition on a general superfield

$$\mathrm{d}^m\phi = 0 \,. \tag{4.7}$$

In the case of SU(4) one can impose a further constraint

$$\mathrm{d}^m\mathrm{d}^n\bar{\phi} = \tfrac{1}{2}\epsilon^{mnpq}\bar{\mathrm{d}}_p\bar{\mathrm{d}}_q\phi \,. \tag{4.8}$$

A scalar superfield satisfying (4.7) and (4.8) can now be written as

$$\begin{aligned}\phi(x,\theta) = {} & \frac{1}{\partial^+}A(y) + \frac{i}{\partial^+}\theta^m\bar{\chi}_m(y) + \sqrt{\tfrac{1}{2}}i\theta^m\theta^n\bar{C}_{mn}(y) \\ & + \tfrac{1}{6}\sqrt{2}\theta^m\theta^n\theta^p\epsilon_{mnpq}\chi^q(y) + \tfrac{1}{12}\theta^m\theta^n\theta^p\theta^q\epsilon_{mnpq}\partial^+\bar{A}(y) \,,\end{aligned} \tag{4.9}$$

with $y = (x, \bar{x}, x^+, x^- - \sqrt{\tfrac{1}{2}}i\theta^m\bar{\theta}_m)$, where the overall factor $(\partial^+)^{-1}$ has been inserted for convenience. Note that this superfield contains only component fields of canonical dimension. This is automatic in the construction due to the supersymmetry algebra. Furthermore we see that the expansion in θ also means an expansion in the different helicity components starting with $+1$ for $A(x)$ decreasing by $\frac{1}{2}$ for each power of θ.

It is now straightforward but tedious to rewrite the action (3.13) in terms of this superfield. The result is

$$\begin{aligned}S = 72\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\Big\{ & -\bar{\phi}^a\frac{\Box}{\partial^{+2}}\phi^a \\ & + \tfrac{4}{3}gf^{abc}\left(\frac{1}{\partial^+}\bar{\phi}^a\phi^b\bar{\partial}\phi^c + \text{complex conjugate}\right) \\ & - g^2f^{abc}f^{ade}\left(\frac{1}{\partial^+}(\phi^b\partial^+\phi^c)\frac{1}{\partial^+}(\bar{\phi}^d\partial^+\bar{\phi}^e) + \tfrac{1}{2}\phi^b\bar{\phi}^c\phi^d\bar{\phi}^e\right)\Big\} \,,\end{aligned} \tag{4.10}$$

where $\mathrm{d}^4\theta$ is normalized so that $\int \mathrm{d}^4\theta\,\theta^4 = 1$.

This action was obtained by a direct comparison with (3.13). Just considering the action (4.10) the different terms look quite arbitrary. Each term is obviously invariant under restricted supersymmetry. However, although this action is not

Lorentz invariant, the resulting equations of motion should be and this restricts the possible terms in the action. Thus Lorentz invariance still governs the form of the action. By properly deriving the equations of motion from the action (4.10) and properly defining the Lorentz transformation on ϕ one can in principle check the Lorentz invariance of the theory.

Light-cone supersymmetry and superfields have already been discussed by Siegel and Gates [12]. They treated the $N=1$ Wess–Zumino and $N=1$ Yang–Mills theories and showed how to write the free actions only in terms of light-cone superfields. They also gave a general procedure for obtaining these superfields from the usual superspace.

In a forthcoming paper we will derive the Feynman rules for the superfield ϕ. Although this superfield is constrained, functional differentiations with respect to it can be consistently defined. Hence we have found a formalism where both the SU(4) invariance and the restricted supersymmetry are manifest and we believe this will simplify quantum calculations to the extent that the question of finiteness can be resolved.

5. On the light-cone formulation of other supersymmetric theories

We have shown so far that the $N=4$ Yang–Mills field can be described in terms of a scalar superfield. It is easily seen that the same is true for all supersymmetric Yang–Mills theories. For each N we use a θ^m in the N-dimensional representation. For all N's but $N=4$ the superfield is chiral but otherwise unconstrained. E.g. for $N=1$ the superfield is $(y=(x,\bar{x},x^+,x^- - \frac{1}{2}i\theta\bar{\theta}))$

$$\phi(x,\theta)=A(y)+i\theta\bar{\chi}(y)\,, \tag{5.1}$$

$$\bar{\phi}(x,\theta)=\bar{A}(y)+i\bar{\theta}\chi(y)\,. \tag{5.2}$$

It is straightforward to rewrite the light-cone action for this theory in terms of this superfield.

The $N=3$ and $N=4$ theories have the same particle content. Hence they will be equivalent in the light-cone formulation and we have a choice which superfield to use. If we use the $N=3$ formulation we have the advantage of using an unconstrained chiral superfield, but lose the explicit SU(4) invariance. Since, however, the variation with respect to $N=4$ superfield is fairly simple to perform although it is constrained we feel that this formalism is the more advantageous one and we will use it in the sequel.

The light-cone formalism should also be applicable to supergravity theories. Since both the spin-2 and the spin-1 particles have two helicity states they can be described by complex scalar fields. Similarly the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ particles each have two helicity states and can be described by scalar complex Grassmann fields. Only the spin-0 fields represent one degree of freedom and hence must have extra constraints. This is most easily achieved by having those fields just in the middle of a chiral superfield so that they can be made to satisfy a duality constraint as in the $N=4$ Yang-Mills theory. Hence the $N=8$ supergravity fields can all be contained in a scalar chiral superfield with a θ^m transforming as an 8 under SU(8). We write it as

$$\phi(x_1\theta)=\frac{1}{\partial^{+2}}h(y)+i\theta^m\frac{1}{\partial^{+2}}\bar{\psi}_m(y)+\tfrac{1}{2}i\theta^m\theta^n\frac{1}{\partial^+}\bar{A}_{mn}(y)$$
$$-\frac{1}{3!}\theta^m\theta^n\theta^p\frac{1}{\partial^+}\bar{\chi}_{mnp}(y)-\frac{1}{4!}\theta^m\theta^n\theta^p\theta^q\bar{C}_{mnpq}(y)$$
$$+\frac{i}{5!}\theta^m\theta^n\theta^p\theta^q\theta^r\epsilon_{mnpqrstu}\chi^{stu}(y)$$
$$+\frac{i}{6!}\theta^m\theta^n\theta^p\theta^q\theta^r\theta^s\epsilon_{mnpqrstu}\partial^+A^{tu}(y)$$
$$+\frac{1}{7!}\theta^m\theta^n\theta^p\theta^q\theta^r\theta^s\theta^t\epsilon_{mnpqrstu}\partial^+\psi^u(y)$$
$$+\frac{1}{8!}\theta^m\theta^n\theta^p\theta^q\theta^r\theta^s\theta^t\theta^u\epsilon_{mnpqrstu}\partial^{+2}\bar{h}(y)\,, \tag{5.3}$$

with $y=(x,\bar{x},x^+,x^--\frac{1}{2}i\theta^m\bar{\theta}_m)$.

The field components represent the different helicity components of $N=8$ supergravity starting with helicity 2 for $h(x)$ and the general field component has helicity $2-\frac{1}{2}n$ where n is the number of θ's multiplying the field component in the expansion. The fact that this superfield also satisfies a duality constraint such as (4.8) is characteristic of theories with maximal supersymmetry and gives rise to the typical mirror structure of the superfield. It is most likely that this constraint is the property responsible for the unique quantum behaviour that these theories seem to exhibit.

The $N=8$ supergravity is a great challenge to construct in this framework. Since this theory contains non-polynomial interactions one must develop suitable techniques to construct actions in superspace. Furthermore the Lorentz invariance will be more difficult to recover for this model, since a great number of field components will have to be integrated out. However, once these problems have been solved, we believe this is an adequate formalism to analyze the question whether $N=8$ supergravity is a finite quantum theory or not.

Appendix

NOTATIONS AND CONVENTIONS

We use a space-like metric for which

$$A^\mu B_\mu=A_iB_i+A_{\mathrm{L}}B_{\mathrm{L}}-A_0B_0$$
$$=A_iB_i-A_+B_--A_-B_+\,, \tag{A.1}$$

where i labels transverse directions and L and 0 the longitudinal and time directions, respectively. The light-cone coordinates are given by

$$x^{\pm}=\sqrt{\tfrac{1}{2}}(x^0 \pm x^3)\,. \tag{A.2}$$

We use the Dirac algebra (note the minus sign)

$$\{\gamma^{\mu}, \gamma^{\nu}\} = -2\eta^{\mu\nu}\,. \tag{A.3}$$

We write explicitly the gauge group indices which are small letters in the beginning of the alphabet, $a, b, c \dots$.

For the internal SU(4) invariance we use indices which are small letters starting from m in the alphabet. Complex conjugation raises and lowers indices and the complex conjugate is denoted by a bar.

Spinor indices are denoted by small Greek letters taken from the beginning of the alphabet. A Majorana spinor is defined by

$$\bar{\lambda}^{\alpha} C_{\alpha\beta} = \lambda_{\beta}\,, \qquad \bar{\lambda} = \lambda^{+}\gamma^{0}\,, \tag{A.4}$$

where C is the charge conjugation matrix, whereas a (anti-)Weyl spinor satisfies

$$(\gamma^{11})_{\alpha}^{\beta}\lambda_{\beta} = (-)\lambda_{\alpha}\,. \tag{A.5}$$

The γ-matrices are always defined to be properly Weyl-projected.

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