Box 14.1 *(continued)*

c. The text tells one how to read out of such expressions the components of the Riemann curvature tensor; for example here,

$$R^{\hat{1}}{}_{\hat{2}\hat{1}\hat{2}} = -R^{\hat{1}}{}_{\hat{2}\hat{2}\hat{1}} = (-1/r)(d^2r/d\sigma^2) \text{ (coefficients of } \boldsymbol{\omega}^{\hat{1}} \wedge \boldsymbol{\omega}^{\hat{2}} \text{ or } \boldsymbol{\omega}^{\hat{2}} \wedge \boldsymbol{\omega}^{\hat{1}}).$$

d. Generalizing to four dimensions, one understands by $R^{\alpha}{}_{\beta\mu\nu}$ the factor that one has to multiply by three numbers to obtain a fourth. The number obtained is the change (with reversed sign) that takes place in the αth component of a vector when that vector is transported parallel to itself around a closed path, defined, for example, by a parallelogram built from two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The factors that multiply $R^{\alpha}{}_{\beta\mu\nu}$ are (1) the component of the vector $\boldsymbol{A}$ in the βth direction and (2, 3) the $\mu\nu$ component of the extension of the parallelogram, $(u^\mu v^\nu - u^\nu v^\mu)$. Thus

$$\delta A^{\alpha} = -R^{\alpha}{}_{\beta|\mu\nu|}A^{\beta}(u^\mu v^\nu - u^\nu v^\mu).$$

Box 14.2 STRAIGHTFORWARD CURVATURE COMPUTATION
(Illustrated for a Globe)

The elementary and universally applicable method for computing the components $R^{\mu}{}_{\nu\alpha\beta}$ of the Riemann curvature tensor starts from the metric components $g_{\mu\nu}$ in a coordinate basis, and proceeds by the following scheme:

$$g_{\mu\nu} \xrightarrow{\Gamma \sim \partial g} \Gamma_{\mu\alpha\beta} \longrightarrow \Gamma^{\mu}{}_{\alpha\beta} \xrightarrow{R \sim \partial\Gamma + \Gamma\Gamma} R^{\mu}{}_{\nu\alpha\beta}.$$

The formulas required for these three steps are

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right), \tag{1}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = g^{\mu\nu}\Gamma_{\nu\alpha\beta}, \tag{2}$$

and

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\partial \Gamma^{\mu}{}_{\nu\beta}}{\partial x^\alpha} - \frac{\partial \Gamma^{\mu}{}_{\nu\alpha}}{\partial x^\beta} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha}. \tag{3}$$

The metric of the two-dimensional surface of a sphere of radius a is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4}$$

To compute the curvature by the standard method, use the formula for ds^2 as a table of g_{kl} values. It shows that $g_{\theta\theta} = a^2$, $g_{\theta\phi} = 0$, $g_{\phi\phi} = a^2\sin^2\theta$. Compute the six possible different $\Gamma_{jkl} = \Gamma_{jlk}$ (there will be 40 in four dimensions) from formula

(1). Thus

$$\begin{aligned} \Gamma_{\theta\phi\phi} &= -a^2\sin\theta\cos\theta = -\Gamma_{\phi\phi\theta}, \\ \Gamma_{\theta\theta\theta} &= \Gamma_{\phi\phi\phi} = 0, \\ \Gamma_{\theta\theta\phi} &= \Gamma_{\phi\theta\theta} = 0. \end{aligned} \tag{5}$$

Raise the first index:

$$\begin{aligned} \Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^{\phi}{}_{\phi\theta} &= \cot\theta, \\ \Gamma^{\theta}{}_{\theta\theta} &= \Gamma^{\theta}{}_{\theta\phi} = 0 = \Gamma^{\phi}{}_{\theta\theta} = \Gamma^{\phi}{}_{\phi\phi}. \end{aligned} \tag{6}$$

Choose a suitable curvature component (one that is not automatically zero by reason of the elementary symmetry $R_{\mu\nu\alpha\beta} = R_{[\mu\nu][\alpha\beta]}$, nor previously computed in another form, as by $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$). In this two-dimensional example, there is only one choice (compared to 21 such computations in four dimensions); it is

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \frac{\partial \Gamma^{\theta}{}_{\phi\phi}}{\partial\theta} - \frac{\partial \Gamma^{\theta}{}_{\phi\theta}}{\partial\phi} + \Gamma^{\theta}{}_{k\theta}\Gamma^{k}{}_{\phi\phi} - \Gamma^{\theta}{}_{k\phi}\Gamma^{k}{}_{\phi\theta} \\ &= \frac{\partial \Gamma^{\theta}{}_{\phi\phi}}{\partial\theta} - 0 + 0 - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} \\ &= \sin^2\theta - \cos^2\theta + \sin\theta\cos\theta\cot\theta; \end{aligned}$$

so

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \tag{7}$$

or

$$R^{\theta\phi}{}_{\theta\phi} = \frac{1}{a^2}. \tag{8}$$

Contraction gives the components of the Ricci tensor,

$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = \frac{1}{a^2}, \qquad R^{\theta}{}_{\phi} = 0, \tag{9}$$

and further contraction gives the curvature scalar

$$R = 2/a^2. \tag{10}$$

A convenient orthonormal frame in this manifold is

$$\boldsymbol{\omega}^{\hat{\theta}} = a\,\boldsymbol{d}\theta, \qquad \boldsymbol{\omega}^{\hat{\phi}} = a\sin\theta\,\boldsymbol{d}\phi. \tag{11}$$

More generally one writes $\boldsymbol{\omega}^{\hat{\alpha}} = L^{\hat{\alpha}}{}_{\beta}\,\boldsymbol{d}x^{\beta}$. To transform the curvature tensor to orthonormal components in this simple but illuminating example of a diagonal metric requires a single normalization factor for each index on a tensor. Thus $v^{\hat{\theta}} = av^{\theta}$, $v^{\hat{\phi}} = a\sin\theta\, v^{\phi}$, $v_{\hat{\theta}} = a^{-1}v_{\theta}$, $v_{\hat{\phi}} = (a\sin\theta)^{-1}v_{\phi}$. Similarly, from $R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$ one finds the components of the curvature tensor,

$$R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2} = R^{\hat{\theta}\hat{\phi}}{}_{\hat{\theta}\hat{\phi}}, \tag{12}$$

in the orthonormal frame.