PRINCIPLES OF QUANTUM MECHANICS

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Contents

1 Wave-Particle Duality **1**

1.1 Interference, waves and particles 2

1.1.1 An experiment with two slits: classical waves 3

1.1.2 An experiment with two slits: classical particles 5

1.1.3 An experiment with two slits: electrons 6

1.1.4 Photons 7

1.1.5 De Broglie wavelength 8

1.2 Experiments 10

1.2.1 Photoelectric effect 10

1.2.2 Compton effect 12

1.2.3 Crystal diffraction 13

1.2.4 A delayed choice experiment 16

1.3 The uncertainty principle 16

Chapter 1

Wave-Particle Duality

In classical mechanics waves and particles are distinct entities. Particles are lumps of energy and momentum that can be specified by coordinate q and velocity $\dot{q}$, or equivalently by coordinate and momentum. Given the initial values of q and $\dot{q}$, the trajectory $q(t)$ is known from Newton's laws. A wave, in contrast, is a disturbance that is extended in space, described by a "wavefunction" $\psi(x,t)$, which may be the excess air pressure for a sound wave, the height of the water for a wave on the surface of water, etc. An electromagnetic wave is the oscillation of electric and magnetic fields, which are vectors, $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$, in contrast to a scalar wave $\psi(x,t)$, but this vector character is not important for the present discussion. A scalar wave is governed by the wave equation

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2}, \tag{1.1}$$

where v is the velocity of the wave moving in the x direction. Given $\psi(x,0)$ and $\dot{\psi}(x,0)$ at the initial time $t=0$, we can predict $\psi(x,t)$ for all times. Classical waves can exhibit interference phenomena, while classical particles can not.

In quantum theory, each particle is associated with a wavefunction $\psi(x,t)$, which is, in general, complex; that is, it has both a real and an imaginary part. In contrast, the wavefunction of a classical wave can always be chosen to be real because it represents the tangible physical world around us. In this case, the use of a complex function is simply a mathematical convenience, not a necessity. We shall learn that in quantum theory the absolute square, $|\psi(x,t)|^2$, is proportional to the probability of finding a particle at x and at time t, and when we do find it, we find it as a particle. A particle may behave as an extended wave or a localized lump depending on the physical situation. That an electron is *both* a wave *and* a particle (or neither a wave nor a particle!) is the wave-particle duality, for which our language, based on our common sense experience, is not adequate. Quantum mechanics is the dynamics of the wavefunction $\psi(x,t)$, which obeys an equation known as the Schrödinger equation. The wave-particle duality implies that an electron must exhibit interference, and light must exhibit properties of particles. The particles of light are the photons.

1.1 Interference, waves and particles

A monochromatic (single frequency) plane wave travelling in the x direction is given by the wavefunction

$$\begin{aligned} \psi(x,t) &= Ae^{i\phi(x,t)}, & (1.2)\\ &= Ae^{i(\frac{2\pi}{\lambda}x-\frac{2\pi}{T}t)}, & (1.3)\\ &= Ae^{i(kx-\omega t)}. & (1.4) \end{aligned}$$

We have three equivalent ways of describing the wave. First, the wavefunction being a complex function must be expressible in terms of a magnitude, which is the real amplitude A and a phase, which is $\phi(x,t)$, a function of position and time. In the second line the phase $\phi(x,t)$ is written in terms of the wavelength λ and the time period T, and in the third, we have introduced the wavevector $k = 2\pi/\lambda$ (actually it's a scalar for this one-dimensional case, but this abuse of language is perhaps forgivable) and the frequency $\omega = 2\pi/T$.

The magnitude of the wavevector k measures the change of phase per unit length at fixed t, and ω measures the change of phase per unit time at a fixed x. This wave is travelling with the wave speed $v = \omega/k$. If we move along x such that $x = (\omega/k)t$, the phase remains unchanged. We will also define the intensity I of a wave by

$$I = |\psi(x,t)|^2. \tag{1.5}$$

For a plane wave this is a constant.

⋆ A plane wave in three dimensions is a simple generalization of the above, replacing the scalar quantities by their vector counterparts. It is

$$\psi(\mathbf{x},t) = Ae^{i(\mathbf{k}\cdot x-\omega t)} \tag{1.6}$$

⋆ The simplest possible spherically symmetric wave in three dimensions moving outward from the origin is

$$\psi(r,t) = \frac{A}{r} e^{i(kr-\omega t)}. \tag{1.7}$$

Unlike the plane wave its amplitude diminishes as the radial coordinate r increases. We hardly need to concern ourselves with this at this stage.

1.1.1 An experiment with two slits: classical waves

Consider a plane wave travelling along the x-direction incident on a screen with two slits as shown in Fig. 1.1. The original wave is diffracted at the

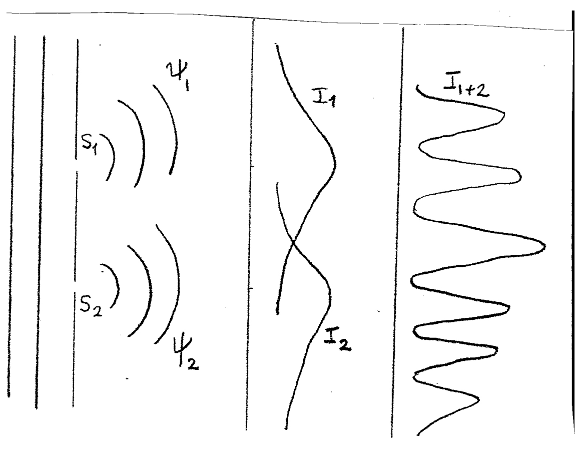

Figure 1.1: Two slit interference pattern for waves.

slits, and new spherical waves spread out to reach a second screen which acts as the detector. If we have only one of the slits open, the intensity will be

$$I_1 = |\psi_1|^2, \quad S_2 \text{ closed}, \tag{1.8}$$
$$I_2 = |\psi_2|^2, \quad S_1 \text{ closed} \tag{1.9}$$

The intensities can have any small value; it does not have the lumpy character of particles. We just have to jiggle the source that produces the original wave less forcefully. Now, what happens when both slits are open? Then, the waves interfere with each other and to calculate the intensity, we first determine the total amplitude, ψ, at any point on the detecting screen and at any given time, as

$$\psi = \psi_1 + \psi_2, \tag{1.10}$$
$$= A_1 e^{i\phi_1} + A_2 e^{i\phi_2} \tag{1.11}$$

The total intensity, I_{1+2} with both slits open is then

$$I_{1+2} = (A_1 e^{i\phi_1} + A_2 e^{i\phi_2})^* (A_1 e^{i\phi_1} + A_2 e^{i\phi_2}), \tag{1.12}$$
$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2). \tag{1.13}$$

So,

$$I_{1+2} \neq I_1 + I_2, \tag{1.14}$$

which is the hallmark of interference. Note that more energy is flowing into the minima of the interference pattern with just one slit open! Opening both slits actually reduces the energy flowing at these minima.

The interference is constructive when the phase difference is an integer multiple of 2π (in phase), that is, $\phi_1 - \phi_2 = 2n\pi$. The total intensity is then given by

$$I_{1+2} = I_1 + I_2 + 2\sqrt{I_1 I_2}, \quad \text{constructive interference} \tag{1.15}$$

Those places where the two waves arrive with a phase difference that is an odd multiple of π, that is, $\phi_1 - \phi_2 = (2n+1)\pi$ (out of phase), the total intensity assumes its minimum value. Then,

$$I_{1+2} = I_1 + I_2 - 2\sqrt{I_1 I_2}, \quad \text{destructive interference} \tag{1.16}$$

At the slits the phase difference between the two waves is zero, that is, they are in phase. A little thought shows that at the location of the constructive interference, the path difference between the waves is a multiple $n\lambda$. Similarly, for destructive interference, the path difference must be $(2n+1)\lambda/2$.

⋆ We astutely considered the case where the phase difference of the two sources (the slits) emitting the interfering waves was constant, zero, in particular (Why?). If, instead, this phase difference were random, there would be no interference pattern (Why?).

⋆ If you carefully look at Fig. 1.1, you will notice that the intensity is drawn to decrease away from the center of the detecting screen. The reason is that the amplitude of a spherical wave decreases with distance, see Eq. 1.1

1.1.2 An experiment with two slits: classical particles

Consider now an experiment with classical particles with the same fixed energy. Imagine that the source that sends out the particles in the x-direction is not perfect. It sprays the particles over a fairly large angular spread as shown in Fig. 1.2. The detector records the probability, P, that the

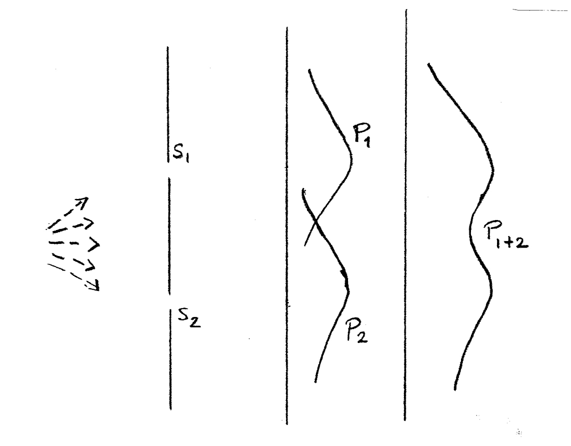

Figure 1.2: Two slit interference pattern for classical particles.

particles get to it. Assuming that the source emits at a constant rate, the probability is simply proportional to the number of hits at the detector during a standard time interval, sufficiently large compared to the time between the successive emissions of the particles. This time the intensity of the hits at the detector cannot have any small value, for the classical particles are lumps of momentum and energy, arriving at the detector one at a time. Therefore, the intensity has a grainy character. The crucial point is that each particle travels through either the slit S_1 *or* the slit S_2. The laws of Newton do not allow for anything different. The trajectories are uniquely determined by the initial conditions. Although these initial conditions are random, there is a group that comes through S_1 and a group through S_2. It follows therefore that the probability with both slits open is the sum of the probabilities with the individual slits open separately, that is,

$$P_{1+2} = P_1 + P_2. \tag{1.17}$$

This equation says that the total probability is simply the sum of the two probabilities, that is, there is no interference for classical particles.

1.1.3 An experiment with two slits: electrons

We now do the same experiment with monoenergetic electrons shown in Fig. 1.3. The detector records the arrival of the individual electrons as discrete events, just as we expect for particles. We can again take the intensity at the detector as the measure of the relative probability, that is, the chance that an electron arrives at the detector. With only one of the slits open, the probability as a function of position on the screen looks just like what our classical common sense would predict. But with both slits open, surprisingly, we find that the total probability, P_{1+2} is no longer the sum of the individual probabilities, P_1 and P_2. Instead, it is characteristic of the interference phenomena observed for waves. We conclude that the electrons behave like

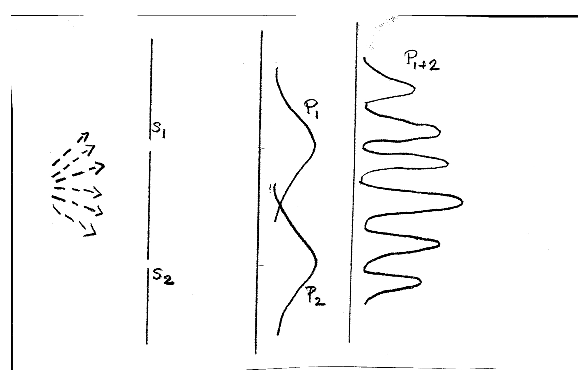

Figure 1.3: Two slit interference pattern for electrons.

particles insofar as their interaction with the detector is concerned, but at the same time they behave like waves because they produce an interference pattern. This behavior of an electron can be understood if we associate a complex wavefunction $\psi(x,t)$ with it and interpret $|\psi(x,t)|^2$ to be proportional to the probability of finding an electron at x and t. We shall refer to ψ as the probability amplitude.

When an electron can get to (x,t) in more than one way, we must first sum the probability amplitudes and then square it to find the total probability, otherwise we would loose an important quantum effect—interference. In general, if an event can occur in alternative ways, we must first sum the alternate amplitudes and then square the sum to find the probability of the event. This, we shall call the **principle of superposition**.

We are forced to give up the notion of a trajectory, for if it were valid, an electron would come through either S_1 or S_2, thereby destroying the interference pattern. This wave-particle duality of an electron is the fundamental mystery of quantum mechanics. Nobody or no experiments have ever found a way out. You might try to check this more deeply by observing which slit the electron goes through and correlating that information with the probability pattern seen at the detector. Surely, we can find this out if we set up a source of light close to the slits so that it produces a flash near the slit in question. Careful arguments show that if we succeed in decoding which slit it goes through, we also destroy the interference pattern, and the total probability becomes simply the sum of two individual probabilities, same as classical particles.

⋆ For recent experiments demonstrating interference effects with electrons, see A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, *American Journal of Physics*, **57**, 117 (1989). For earlier experiments, see the translation of Claus Jönsson's article on electron two-slit interference experiment in *American Journal of Physics* **42**, 4 (1974).

1.1.4 Photons

Finally, we can repeat the two-slit interference experiment with light, that is, electromagnetic waves. A electromagnetic wave is described by the wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.18}$$

for the electric field $\mathbf{E}(\mathbf{x}, t)$ and a similar equation for the magnetic field $\mathbf{B}(\mathbf{x}, t)$ (What is it?). The velocity of light is denoted by c, and the Laplacian by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. A propagating plane wave is given by the real part of the oscillating electromagnetic wave

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{1.19}$$

When substituted in the wave equation it gives the relation

$$\omega = ck = 2\pi c/\lambda, \tag{1.20}$$

where k is the magnitude of the wavevector. We also know that the electric field has to be perpendicular to the direction of $\mathbf{k}$, and the magnetic field has to be perpendicular to both the electric field and the direction of propagation.

By now it will come as no surprise to you that the two-slit experiment will produce an interference pattern, just the way we have described earlier. Now, let's turn the intensity of light down. As we do this, we will discover that the detector is recording discrete bursts, as though light were made of particles. Of course, the interference pattern will eventually build up as we record more and more hits. The particles, or the quanta, of light are the photons, which must move with the velocity of light c. According to the theory of relativity, they must have zero rest mass. Then from the relation between the energy, E, and the momentum, p, given by

$$E^2 = p^2c^2 + m^2c^4, \tag{1.21}$$

we find that for photons

$$E = pc. \tag{1.22}$$

Of course, we know from Planck's prescription that photons carry energy $E = \hbar\omega$. Thus, the momentum of photon is given by

$$p = E/c = \hbar\omega/c = \hbar k = h/\lambda, \tag{1.23}$$

where $\hbar = h/2\pi = 1.05459 \times 10^{-27}$ erg. sec, h being the Planck's constant. The momentum of a photon is inversely proportional to its wavelength. Einstein was able to explain photoelectric effect from the concept of photons as we shall see in a later section. Another neat effect is the Compton effect in which x-ray photons are scattered from an electron of mass m that is initially at rest. Thus, light has both wave and particle properties. The intensity at the detector at a given location depends on the number of photons arriving at that location. But we know that the intensity is also proportional to the electric field. So, the probability to observe photons is also proportional to the absolute square of the electric field.

Why don't we observe the particle-like behavior of ordinary everyday light. The reason is that such electromagnetic waves contain so many photons that we do not notice the graininess. Actually, the fact that photons are Bosons plays an important role—we shall discuss this much later in the course.

1.1.5 De Broglie wavelength

Louis de Broglie (the most mispronounced name in the world) had an interesting idea that if we can associate a particle of momentum p with a wave of wavelength λ as $p = h/\lambda$ as in the case of light, we could perhaps associate a wave of wavelength λ with a material particle of momentum p (another expression of the concept of duality). So, for a material particle, we have the de Broglie relation

$$p = h/\lambda. \tag{1.24}$$

This has turned out to be spectacularly correct as we shall see as we continue with these lectures. In using Eq. 1.24 do not use the relation $E = \hbar\omega$ for light, for this is an equation for a material particle.

Now we can answer why for a classical particle we do not observe interference. Consider a mass of 1 gm moving with a velocity of 1 cm/s. The de Broglie wavelength for this object is of order 10^{-26} cm! Just try to imagine how small that is. It is 10^{-13} of the size of a proton. With such small wavelengths the two-slit interference pattern will be washed out. A simple calculation shows that the separation between the successive maxima or minima is $\lambda D/d$, where D is the distance between the screen containing the slits and the detector, and d is the separation between the slits. On any reasonable scale, the interference pattern will undergo such unbelievably rapid oscillations that there would be nothing left of it.

• EXERCISE **1.1**

In the two-slit interference experiment with electrons, show that the constructive interference takes place if the path difference of the two waves is an integer multiple, $n\lambda$, of the de Broglie wavelength, λ. Similarly, destructive interference takes place when the path difference is $(2n + 1)\lambda/2$.

• EXERCISE **1.2**

Show why the interference pattern is washed out if we do a two-slit experiment with baseballs? Estimate first the de Broglie wavelength, λ, (use some reasonable parameters) and then show that the separation, Δx, between the successive minima or maxima of the interference pattern is $\lambda D/d$, where D is the distance between the screen containing the slits and the detector, and d is the separation between the slits. Find Δx; what does it imply for the interference pattern? Explain carefully.

• EXERCISE **1.3**

(a) What is the wavelength of a X-ray photon of energy 1 keV? (b) What is the wavelength of a gamma-ray photon of energy 1 MeV? (c) What is the range of energies of photons of visible light?

• EXERCISE **1.4**

(a) Consider thermal neutrons at the room temperature, whose average kinetic energy is $\frac{3}{2}k_B T$, where k_B is the Boltzmann's constant and T is the temperature.

What's the de Broglie wavelength? (b) In large accelerators you can produce beams of electrons of very high energy. What is the de Broglie wavelength of an electron of energy 1 Gev?[Be careful to use the relativistic formula relating the energy and the momentum.]

• EXERCISE **1.5**

In the two-slit interference experiment of R. Gahler and A. Zellinger, *American Journal of Physics* **59**, 316 (1991), the beam of neutrons had a kinetic energy of 2.4×10^{-4} eV. (a) What is the de Broglie wavelength of neutrons with this kinetic energy? (b) Estimate the de Broglie wavelength from the interference pattern shown in the figure taken from the actual experiment and compare with that obtained in part (a). The distance from the slits to the detector is 5 m and the separation between slits is 1.26×10^{-2} cm.

1.2 Experiments

In this section we shall discuss the key experiments that support wave-particle duality, which have been historically important in the development of quantum mechanics, but the emphasis here is not historical at all. We therefore do not discuss many wrong attempts to explain these experiments. History of physics is an important and serious discipline, but has no place in a textbook of this kind.

1.2.1 Photoelectric effect

Light shining on a metal can eject electrons, which is called the photoelectric effect. The correct explanation due to Einstein is that a single photon of energy $\hbar\omega$ delivers all its energy to an electron, and if the energy of the photon is greater than the work function of the metal, Φ, the electron will be ejected with the kinetic energy E. But because the electron can loose its energy by other processes as it reaches the surface, we only have the inequality

$$E \leq \hbar\omega - \Phi. \tag{1.25}$$

The maximum energy with which the electron can be emitted is evidently

$$E_{\text{max}} \leq \hbar\omega - \Phi. \tag{1.26}$$

The work function is the energy required to extract an electron from a metal. Clearly, it must be a non-zero amount, otherwise the electrons will boil out of it. The magnitude of the work function is of order a few eV. As an example, the work function of a so called transition metal Ni is 4.6 eV, while that of an alkali

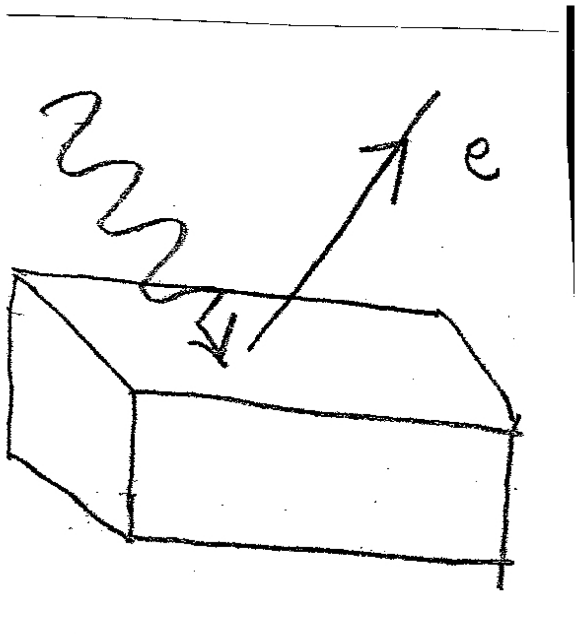

Figure 1.4: The photoelectric effect.

metal such as Cs is 1.8 eV. For a given metal, the minimum energy necessary for a photon to eject a photoelectron must be greater than or equal to the work function. If it has more energy, the rest goes to the kinetic energy of the outgoing electron. Remembering the relation between the wavelength and the energy of a photon, you can see that the wavelength of a 4 eV photon is 3100 Å, that is in the ultraviolet.

There are three important characteristics of the photoelectric effect none of which can be explained by the classical theory of light.

1. The maximum kinetic energy is independent of the intensity of light. The intensity simply determines the number of ejected electrons.

2. For a given metal, light of a minimum frequency is necessary.

3. The effect takes place virtually instantaneously, within about 10^{-9} s.

It is not difficult to see why the classical theory predicts incorrect results. Greater intensity would imply greater energy delivered to the metal and hence the electrons. Similarly, if the light is intense enough, the photoelectric effect should take place no matter what the frequency or the wavelength of light is. The minimum time is about the minimum energy necessary to eject an electron divided by the average power, which is the intensity I times the area, A, of the target, say about the size of an atom. This is a number of order seconds.

In the modern days, a sophisticated version of the photoelectric effect is used as a spectroscopy to analyze the quantum states of the electrons inside a metal. The idea is very simple. One measures both the kinetic energy the momentum of the outgoing electron, which enables one, using the laws of energy and momentum conservation, to deduce the energy and the momentum of the quantum state in the metal the electron came from.

1.2.2 Compton effect

The photoelectric effect is an absorption process in which a photon is absorbed and an electron is released from a metal, while the Compton effect is a scattering process in which an incident photon scatters from a free electron. The photon scatters as if it were a particle of energy $E = \hbar\omega$ and a momentum of magnitude $p = E/c$. In Compton effect, not only are we exploring the energy-frequency relation, $E = \hbar\omega$, of a photon, but also its momentum as expressed as $p = E/c$. Typically, X-ray photons are scattered from a target such as graphite. The energy of these photons are so high that they liberate the electrons from the atoms, and, in these circumstances, one can assume that the scattering is from free electrons. A hierarchy of physical processes take place as we increase the photon energy; for

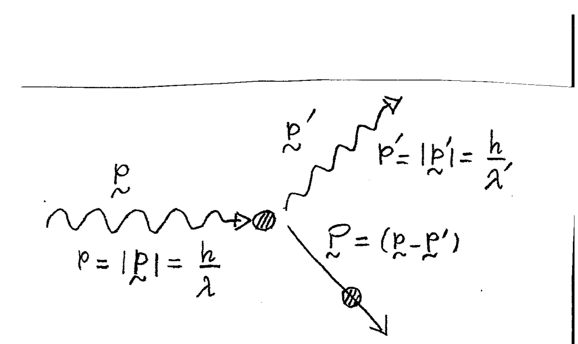

Figure 1.5: The Compton effect.

energies less than about 500 keV, we have the photoelectric effect, while between 500 keV and 5 Mev we have the Compton effect. For even higher energies we have the electron-positron pair production, where a positron is the antiparticle of an electron.

The photons move at the speed of light, so we must use relativistic kinematics. Before the collision, we have a photon moving towards the electron with a momentum $\mathbf{p}$ and an energy pc. After the collision, the photon flies off with a momentum $\mathbf{p}'$ and the electron with a momentum $(\mathbf{p} - \mathbf{p}')$ from the conservation of momentum. The scattering angle θ is the angle between the incident photon momentum and the final photon momentum. The energy conservation gives

$$pc + mc^2 = p'c + \left[(\mathbf{p} - \mathbf{p}')^2 c^2 + m^2 c^4\right]^{1/2} \tag{1.27}$$

Now move $p'c$ to the left hand side and square both sides. With a little manipulation, you can derive

$$p' = \frac{mcp}{mc + p(1 - \cos\theta)}. \tag{1.28}$$

Now using the relation between the magnitude of momentum and the energy of photon, we have the relation

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta) \geq 0. \tag{1.29}$$

The minimum shift of the wavelength is at $\theta = 0$, while the maximum shift is at $\theta = \pi$, corresponding to

$$(\lambda' - \lambda)_{\max} = 2\frac{h}{mc} = 0.048\text{Å} = 4.8 \times 10^{-10}\text{cm}. \tag{1.30}$$

The quantity (h/mc) is called the Compton wavelength of an electron. This relationship is precisely verified in experiments.

If the energy of the incident photon is not sufficiently large, the electron cannot be ejected. In this case, the entire atom recoils as a whole, and because it is typically 10^4 heavier than the electron (You must substitute the mass of the atom for the mass of the electron in the above formula.), the Compton shift will be immeasurably small.

1.2.3 Crystal diffraction

While discussing de Broglie wavelength, we noted that for the two-slit interference pattern to be observable an electron must have a wavelength comparable to the separation between the slits. This is also true for electron diffraction, which is another manifestation of its wave nature. The Nature provides us with such a situation in a crystal where the atoms are located on a regular crystalline array of spacing of a few Å. Such an array will diffract X-rays, electrons, neutrons, or whatever if the wavelength is right and each atom is capable of elastically scattering these waves; we shall consider only elastic scattering, that is, without any change of energy. One can easily produce electron beams of such wavelengths by accelerating electrons under a potential drop, and observe the diffraction pattern as the electrons are reflected from the crystal.

To understand electron diffraction, consider two *adjacent* planes of atoms in a crystal as shown in Fig. 1.6. Assume that the incident waves are reflected specularly (mirror-like) from each plane, with each plane reflecting a small fraction of the incident beam, like a partially silvered mirror.

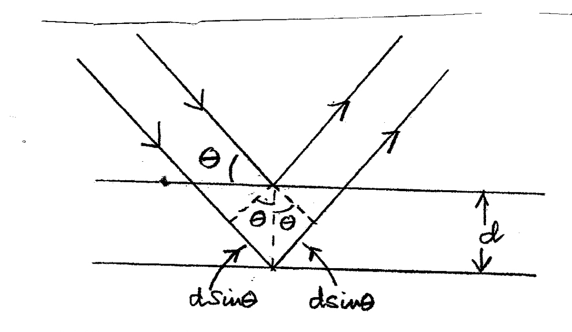

Figure 1.6: Crystal diffraction.

There will be constructive interference, and a strong reflection, when the path difference satisfies the relation

$$2d\sin\theta = n\lambda, \tag{1.31}$$

where $n = 1, 2, \ldots$. This is known as the Bragg's law. It is clear that $2d$ must be greater than or equal to $n\lambda$, for $|\sin\theta| \leq 1$.

The reflection from all parallel planes add up in phase to give a strong reflection. Because the electrons are charged particles, they cannot penetrate very far into the crystal without loosing energy, and only a small number of layers close to the surface contribute to the diffracted beam. Thus, the electron diffraction experiment will produce broader maxima than X-ray diffraction, because X-rays can penetrate deep into the bulk and about 10^3 to 10^5 layers constructively contribute to the diffracted beam.

We have taken a simplified view of a crystal in which a single atom is repeated in a particular pattern defining the type of the lattice. A real crystal is often made of a repetition of a basic unit shown in Fig. 1.7. Such a repetition unit is called a basis and a crystal structure is the underlying lattice plus the basis and can be constructed by repeating the unit cell to fill the volume. The lattice type can be determined by examining the symmetry of the reflection, but to find the basis we must analyze the intensity of the diffracted beam at various angles.

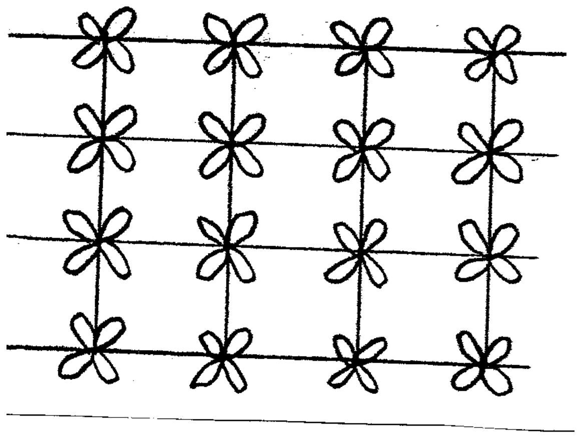

Figure 1.7: A lattice with a basis.

A crystal can be partitioned into many different sets of parallel planes as shown in Fig. 1.8. These sets of parallel planes will diffract a incident wave at various angles θ_i corresponding to the separations between the planes d_i. For a given wavelength of the incident wave, it may be difficult to find the correct set of planes and the corresponding angles. On the other hand for a beam consisting of a continuous range of wavelengths, each d_i and θ_i will lead to a strong diffraction for the wavelength λ_i. The diffraction pattern thus generated has the symmetry of the crystal structure and is known as the Laue pattern.

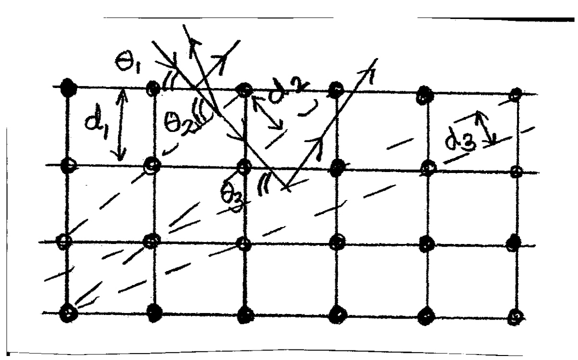

Figure 1.8: A crystal partitioned into various sets of lattice planes.

1.2.4 A delayed choice experiment

In an article with the title "The Reality of the Quantum World" [*Scientific American* **258**, 46 (January, 1988)], Abner Shimony discusses a recent experiment to reveal the strangeness of the quantum world. A photon from a laser is incident on a beam splitter. The two components can be recombined, or a switch in one of the paths can be turned on to detect if the photon has taken that particular path. The laser is of such low intensity that there is only one photon at any given time in the apparatus. Two questions pertinent to the notion of wave-particle duality are: (1) Does the photon take a definite path such that it is either transmitted or reflected, by exhibiting particle-like property? (2) Or is the photon in some sense both transmitted and reflected and interferes with itself, exhibiting its wavelike property? If the switch is on, the photon may be detected by a photodetector, thereby answering the question which path it took. If the switch is off, the photon is free to interfere with itself and produce an interference pattern like a wave. The results show that that a photon behaves like a particle if particlelike properties are measured and behaves like a wave if its wavelike properties are measured. Using a special fast switch (with a response time far shorter than the time it takes the photon to traverse the apparatus), the switch was triggered after the photon had interacted with the beam splitter, so that the photon could not be informed whether to behave like a particle or a wave. The experiment is a delayed choice experiment, because the choice of what kind of experiment to do was made after the photon interacted with the beam splitter.

1.3 The uncertainty principle

Here is a first look at the uncertainty principle to which we shall return many times during these lectures. In fact, we have already seen a version of it when we noted that we could not find out which path an electron took in a two-slit interference experiment without destroying the interference pattern (With a bit of effort you should be able to convince yourself of this.). We are now going to desribe an experiment that is easy to understand and shows the relationship between the uncertainties of position and momentum.

Imagine a beam of electrons, all of the same energy and momentum, p_x, incident on a screen with a slit whose width is Δy. Before an electron passes through the slit, its vertical position is completely uncertain (being a wave), but its vertical momentum is precisely known to be zero. An electron gets diffracted at the slit, and, after passing through the slit, we know its vertical position with an uncertainty Δy. Now, there is a spread in the vertical component of its momentum, which we shall call Δp_y, because there is a spreading of the wave at the slit.

We can estimate Δp_y from the single slit diffraction pattern shown in Fig. 1.9. If $\Delta\theta$ is the angle at which the first minimum of the pattern occurs, then $\Delta p_y =$

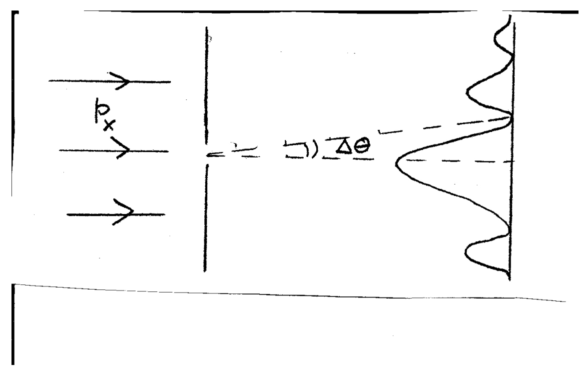

Figure 1.9: Diffraction from a single slit illustrating the uncertainty relation.

$p_x \Delta\theta$. For simplicity we shall assume that the screen is so far away from the slit that $\sin\Delta\theta = \tan\Delta\theta \approx \Delta\theta$. But what is $\Delta\theta$? From the theory of diffraction of a wave at a slit, we know that $\Delta\theta = \lambda/\Delta y$. Then,

$$\Delta p_y = p_x \Delta\theta = p_x \frac{\lambda}{\Delta y}, \tag{1.32}$$

which we can combine with the de Broglie relation, $p_x = h/\lambda$, to get the uncertainty relation

$$\Delta p_y \Delta y = h. \tag{1.33}$$

The uncertainty in the vertical momentum, Δp_y, is large if the uncertainty in the vertical position, Δy, is small, and vice versa. The product of the two uncertainties is a fundamental constant of Nature.

• EXERCISE **1.6**

The work function Φ for Aluminum is 4.2 eV. What is the kinetic energy of the fastest photoelectrons if the incident light is of wavelength 1500 Å? What is the kinetic energy of the slowest emitted electrons? What is the average number of photons per unit time per unit area if the intensity of the incident light is 10 W/m^2. What is the smallest energy a photon must have in order to be able to eject an electron from aluminum?

• EXERCISE **1.7**

The intensity of light incident on sodium metal is 100 W/m^2. The work function, ϕ, of sodium is 2.3 eV. Assume that an electron is confined to a circular area of radius 1 Å. Calculate the time it will take the surface to absorb enough energy to release an electron. Note that for this classical estimate the wavelength of light never enters.

• EXERCISE **1.8**

A photon of wavelength 0.024Å is Compton scattered from a free electron. Find the wavelength of a photon which is scattered by 30° from the incident direction? What is the kinetic energy of the outgoing electron?

• EXERCISE **1.9**

Consider a crystal with atoms arranged in a cubic array. The distance between an atom and its nearest neighbor, also known as the lattice spacing, is 0.91Å. If 300 eV electrons are used, at what angle from the crystal surface must they be incident to produce a first order maximum? Consider now Bragg reflection from atomic planes connecting diagonally situated atoms. What is the longest wavelength electrons can have so as to produce a first-order maximum? What is the corresponding kinetic energy?

• EXERCISE **1.10**

This problem should refresh your memory of the single slit diffraction pattern. Show that the angular separation, $\Delta\theta$, between center of the pattern to the first minimum, as shown in the figure, is given by $\sin\Delta\theta = \lambda/a$, where λ is the wavelength and a is the width of slit.